The Riesz-Thorin theorem : proof and applications

Romain Kremeurt, Hadrien Maffioli

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I Introduction

Let T be a linear operator mapping continuously $\mathcal{A}_0$ into $\mathcal{B}_0$ and $\mathcal{A}_1$ into $\mathcal{B}_1$ where $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$ and $\mathcal{B}_1$ are Banach spaces. We can show that there exist an infinite numbers of "intermediate" Banach spaces where T is bounded, which means that T will also be continuous. This is called the theory of interpolation of linear applications that throve during the fifties and the sixties and it has many applications, particularly in Fourier analysis.
The gateway to this section of mathematics is the Riesz-Thorin theorem (also call M. Riesz convexity theorem) whose statement and demonstration will be the main goal of this paper. Indeed it is a very famous theorem and it is at the foundation of complex interpolation. Initially M. Riesz was able to prove the result for specifics pairs of Banach spaces and his student: O. Thorin generalized it to the whole square later.
The first proof was given by M. Riesz in 1926 but it has complicated computational features to it so we will focus on a demonstration of the general result given by O. Thorin in 1939. The idea of O. Thorin rests on a complex analysis theorem called the three lines theorem of Hadamard that we will start by proving. Hence it is clear that this result is based on complex spaces unlike the Marcinkiewicz theorem which is the basis for real interpolation
We will then give a few examples of the Riesz-Thorin theorem applications like the Hausdorff-Young theorem that gives a strong result for the Fourier transform.
Finally, we will explore the theory of interpolation of Banach spaces, and show some results.

II The Three lines theorem

In order to achieve our goal, we must first prove an intermediate theorem, which is the "Three lines theorem". This result is a kind of generalization of the maximum principle, that is to say, given a continuous bounded function $f : S \longrightarrow \mathbb{C}$ analytical on $\mathring{S}$ (where $S = \mathrm{Re}^{-1}([0\,;1])$), it gives an upper bound on S which depends only on the values of f at the border of S.
It states the following :

Theorem : the Three lines theorem

Let $S = \mathrm{Re}^{-1}([0\,;1])$ and $F\colon S \longrightarrow \mathbb{C}$ be a continuous bounded function, analytical on $\mathring{S}$.
Let us suppose there exists $M_0, M_1 \in \mathbb{R}_+$ such that for every $y \in \mathbb{R}$, $|F(iy)| \leq M_0$ and $|F(1+iy)| \leq M_1$ Then :

$$\forall x \in [0\,;1], \forall y \in \mathbb{R}, |F(x+iy)| \leq M_0^{1-x} M_1^x$$

Proof :

Let us consider that $M_0 = M_1 = 1$.
First, let us suppose that $|F(x+iy)| \xrightarrow[|y|\to+\infty]{} 0$ uniformly for $x \in [0\,;1]$. (*)
Then : $\exists y_0 \in \mathbb{R}_+, \forall\,|y| \geq y_0, \forall x \in [0\,;1], |F(x+iy)| \leq 1$.
Moreover, let us consider $\mathcal{R}$ the rectangle of vertices $1 - iy_0, 1 + iy_0, iy_0, -iy_0$ which is a connected bounded open non-empty subset of $\mathbb{C}$.
We have : $\forall z \in \partial\mathcal{R}, |F(z)| \leq 1$.
Thanks to the maximum principle : $\sup_{\mathcal{R}} |F| = \max_{\partial\mathcal{R}} |F| \leq 1$.
We then conclude that for all $z \in S, F(z) \leq 1$.

In the general case, if F doesn't verify (*), we can consider a sequence of functions that does, and get back to this case.

To do so, let us consider $F_n(z) = F(z) \exp\left(\frac{z^2-1}{n}\right)$, where $z = x + iy$.

$$\begin{aligned}
|F_n(z)| &= |F(x+iy)| \left|\exp\left(\frac{x^2+2ixy-y^2-1}{n}\right)\right| \\
&= |F(x+iy)|\, e^{-\frac{y^2}{n}}\, e^{\frac{x^2-1}{n}} \\
&\leq |F(x+iy)|\, e^{-\frac{y^2}{n}} \text{ because } \frac{x^2-1}{n} < 0 \\
&\leq \left(\sup_S |f|\right) e^{-\frac{y^2}{n}} \text{ because } f \text{ is bounded on } S \\
&\xrightarrow[|y|\to+\infty]{} 0 \text{ uniformly for } x \in [0\,;1]
\end{aligned}$$

Furthermore, for all $y \in \mathbb{R}, |F_n(iy)| \leq 1$ and $|F_n(1+iy)| \leq 1$, so we have $|F_n(z)| \leq 1$ for every $n \in \mathbb{N}$ and every $z \in S$.
In conclusion, as n tends to $+\infty$, for every $z \in S$ we have $|F(z)| \leq 1$, which is what we found out earlier.

More generally, if we don't have $M_0 = M_1 = 1$, we can consider the following function (instead of F) :

$$G : \left|\begin{array}{ccc} S & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \dfrac{F(z)}{M_0^{1-z} M_1^z} \end{array}\right.$$

G is still analytical as quotient and composition of analytical functions, and is also bounded. Furthermore, let $y \in \mathbb{R}$:

$$\begin{aligned}
G(iy) &= \frac{F(iy)}{M_0^{1-iy} M_1^{iy}} \\
&= \frac{F(iy)}{M_0 e^{-iy \ln M_0} e^{iy \ln M_1}}
\end{aligned}$$

We then have :

$$\begin{aligned}
|G(iy)| &= \left|\frac{F(iy)}{M_0 e^{-iy \ln M_0} e^{iy \ln M_1}}\right| \\
&= \frac{|F(iy)|}{M_0} \\
&\leq 1
\end{aligned}$$

The computation is the same for $G(1+iy)$, so G verifies the hypothesis of the theorem with $|G(iy)| \leq M_0' = 1$ and $|G(1+iy)| \leq M_1' = 1$.
We then have :

$$\forall x \in [0\,;1], \forall y \in \mathbb{R}, |G(x+iy)| \leq {M_0'}^{1-x} {M_1'}^{x} = 1$$

So, since :

$$\forall x \in [0\,;1], \forall y \in \mathbb{R}, |F(x+iy)| = |G(x+iy)|\, {M_0}^{1-x} {M_1}^{x}$$

we conclude that :

$$\forall x \in [0\,;1], \forall y \in \mathbb{R}, |F(x+iy)| \leq {M_0}^{1-x} {M_1}^{x}$$

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III The Riesz-Thorin theorem

Notation :

From now on let $(X, \mathcal{A}, \mu)$ be a measure space. For all $1 \leqslant p < \infty$, we define $\mathcal{L}^p(X, \mu)$ as the space of measurable functions f such that

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} < \infty$$

For $p = \infty$ we define $\mathcal{L}^\infty(X, \mu)$ as the space of measurable functions such that

$$\|f\|_\infty = \inf\{C \in \mathbb{R}_+ : |f(x)| \leqslant C \text{ for almost every } x\}$$

which means that $\{x : |f(x)| > C\}$ is μ-measurable and has measure zero.
For all $1 \leqslant p \leqslant \infty$ the relation $f = g$ almost everywhere (for the measure μ) defines an equivalence relation.
We can thus define the equivalence class of $f \in \mathcal{L}^p(X, \mu)$ as

$$\{g \in \mathcal{L}^p(X, \mu) : f = g \ \mu\text{-almost everywhere}\}$$

And we now write $L^p(X, \mu)$ or simply $L^p(\mu)$ the set of equivalence classes.
Finally we define $Et(\mu)$ as the space of μ-measurable step functions and we identify a function to its class with respect to equality μ almost everywhere.

Lemma :

Let $(Y, \mathcal{B}, \nu)$ be another measure space and $f \in L^q(\nu)$. We have the following result :

$$\|f\|_q = \sup_{\substack{g \in L^{q^\star} \cap Et(\nu) \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g \rangle|$$

where $\langle f, g \rangle = \displaystyle\int_Y f g \, d\nu$

Proof :

By Hölder inequality, we have that: $\forall g \in L^{q^\star}(\nu)$ such that g is a step function and $\|g\|_{q^\star} \leqslant 1$

$$\begin{aligned} |\langle f, g \rangle| &= \left| \int_Y f \ g \ d\nu \right| \\ &\leqslant \|f\|_q \ \|g\|_{q^\star} \\ &\leqslant \|f\|_q \end{aligned}$$

Thus $\displaystyle\sup_{\substack{g \in L^{q^\star} \cap Et(\nu) \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g \rangle| \leqslant \|f\|_q$.
Moreover any measurable function f is the point-wise limit of a sequence of simple functions.
We can thus find a sequence $(f_n)_n$ of step functions such that $(f_n)_n$ tends to f point-wise.
Moreover we can take $(f_n)_n$ such that: $\forall n \in \mathbb{N}, |f_n| \leq |f|$.

Then we define :
$\forall n \in \mathbb{N},\ g_n : x \longmapsto \dfrac{\overline{f_n}(x)}{|f_n(x)|}\ |f_n(x)|^{q-1}$. Since

$$\begin{aligned} \frac{1}{q}+\frac{1}{q^\star}=1 \quad &\Longrightarrow \quad \frac{q^\star}{q}+1=q^\star \\ &\Longrightarrow \quad q^\star+q=qq^\star \\ &\Longrightarrow \quad qq^\star-q^\star=q \\ &\Longrightarrow \quad (q-1)q^\star=q \end{aligned}$$

we have that

$$\begin{aligned} \|g_n\|_{q^\star} &= \left(\int_\Sigma |(f_n)^{q-1}|^{q^\star} d\nu\right)^{\frac{1}{q^\star}} \\ &= \left(\int_\Sigma |f_n|^q d\nu\right)^{\frac{1}{q^\star}} \\ &= \|f_n\|_q^{\frac{q}{q^\star}} \end{aligned}$$

So for $\widetilde{g_n}= \dfrac{g_n}{\|f_n\|_q^{q-1}}$ we have that: $\|\ \widetilde{g_n}\ \|_{q^\star} = \dfrac{\|f_n\|_q^{\frac{q}{q^\star}}}{\|f_n\|_q^{q-1}}\ = 1$ because

$$\begin{aligned} \frac{1}{q}+\frac{1}{q^\star}=1 \quad &\Longrightarrow \quad 1+\frac{q}{q^\star}=q \\ &\Longrightarrow \quad \frac{q}{q^\star}=q-1 \end{aligned}$$

Moreover,

$$\begin{aligned} |\langle f_n, g_n\rangle| &= \left|\int_\Sigma f_n\ \ |f_n|^{q-1}\ \ \frac{\overline{f_n}}{|f_n|} d\nu\right| \\ &= \left|\int_\Sigma |f_n|^q d\nu\right| \\ &= \|f_n\|_q^q \end{aligned}$$

Thus $|\langle f_n, \widetilde{g_n}\rangle| = \dfrac{\|f_n\|_q^q}{\|f_n\|_q^{q-1}} = \|f_n\|_q$
So by definition of the supremum, since g_n is a step function for all n, we have that :

$$\int_Y |f_n|^q d\nu \leqslant \left(\sup_{\|g\|_{q^\star}\leqslant 1} |\langle f_n, g\rangle|\right)^q \leqslant \left(\sup_{\|g\|_{q^\star}\leqslant 1} |\langle f, g\rangle|\right)^q \text{ since } \forall n \in \mathbb{N}, |f_n| \leqslant |f|$$

But by Fatou's lemma, since $\forall n \in \mathbb{N}, f_n$ is measurable we have that:

$$\int_Y |f|^q d\nu \leqslant \liminf \int_Y |f_n|^q d\nu \leqslant \left(\sup_{\|g\|_{q^\star}\leqslant 1} |\langle f, g\rangle|\right)^q$$

$$\Longrightarrow \|f\|_q \leqslant \sup_{\|g\|_{q^\star}\leqslant 1} |\langle f, g\rangle|$$

It concludes the proof and we do have that :

$$\|f\|_q = \sup_{\substack{g \in L^{q^\star} \cap Et(\nu) \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g\rangle|$$

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Remark :

We can give a simpler proof of this lemma. Indeed we saw in the proof that for all f in $L^q(\nu)$

$$\sup_{\substack{g \in L^{q^\star} \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g\rangle| \leqslant \|f\|_q$$

thanks to Hölder inequality. Morevover we have the equality by applying the equality case in Hölder inequality (the idea is to introduce a function such as g_n in the proof):

$$\sup_{\substack{g \in L^{q^\star} \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g\rangle| = \|f\|_q$$

Then, since $L^{q^\star} \cap Et(\nu)$ is dense in $L^{q^\star}$ and $g \longmapsto |\langle f, g\rangle|$ is continuous we have that:

$$\sup_{\substack{g \in L^{q^\star} \cap Et(\nu) \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g\rangle| = \sup_{\substack{g \in L^{q^\star} \\ \|g\|_{q^\star} \leqslant 1}} |\langle f, g\rangle| = \|f\|_q$$

Notation :

For a linear operator $T : L^p(\mu) \to L^q(\nu)$, if an $M > 0$ such that for all $f \in L^p(\mu)$

$$\|Tf\|_{L^q(\nu)} \leqslant M\|f\|_{L^p(\mu)}$$

exists, we call the smallest M possible the norm of T and we write it $\|T\|_{p\to q}$.

Theorem : The Riesz-Thorin theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Let T be a linear function and we assume that ν is σ-finite. If

$$T : L^{p_0}(\mu) \to L^{q_0}(\nu)$$

is continuous and

$$T : L^{p_1}(\mu) \to L^{q_1}(\nu)$$

is continuous with $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ then for all $\theta \in]0; 1[$

$$T : L^p(\mu) \to L^q(\nu)$$

is continuous for $1 \leqslant p, q \leqslant \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
Moreover, if X and Y are complex spaces we have:

$$\|T\|_{p\to q} \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\|T\|_{p_1\to q_1}^{\theta}$$

and if they are real spaces, we have :

$$\|T\|_{p\to q} \leqslant 2\|T\|_{p_0\to q_0}^{1-\theta}\|T\|_{p_1\to q_1}^{\theta}$$

Remark :

In order to prove this theorem we will use the following result:

Proposition :

If $T : L^p \cap Et(\mu) \to L^q(\nu)$ is continuous then T has a unique continuous extension $\tilde{T} : L^p \to L^q(\nu)$ whose norm is equal to the norm of T.

Proof :

Let T_1 and T_2 be two such extensions of T. For all $f \in L^p$ it exists $(f_n)_{n\in\mathbb{N}} \subset L^p \cap Et(\mu)$ such that $f_n \to f$ in $(L^p, ||.||_p)$. Thus

$$\forall n \in \mathbb{N}, T(f_n) = T_1(f_n) = T_2(f_n)$$

so as n goes to infinity we have $T_1(f) = T_2(f)$ since T_1 and T_2 are continuous. Thus, if such an extension exists, it is unique.
We now assume that such an extension $\tilde{T}$ exists. Then,

$$\forall f \in L^p, \tilde{T}(f) = \lim_{n\to+\infty} \tilde{T}(f_n) = \lim_{n\to+\infty} T(f_n)$$

But, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^p since it converges. Thus,

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall r, s > N,\ \|f_r - f_s\|_p \leqslant \epsilon$$

So since T is linear we have that: $\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall r, s > N,$

$$\begin{aligned}\|Tf_r - Tf_s\|_q &\leqslant \|T(f_r - f_s)\|_q \\ &\leqslant \|T\|_{p\to q} \times \|f_r - f_s\| \\ &\leqslant \|T\|_{p\to q} \times \epsilon\end{aligned}$$

So $(Tf_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^q which is a complete space so it converges (for $\|.\|_q$ norm). Let g be its limit.
In order to show that g does not depend on the choice of $(f_n)_{n\in\mathbb{N}}$, we consider $(\tilde{f}_n)_{n\in\mathbb{N}}$ such that $\tilde{f}_n \to f$ in $(L^p, ||.||_p)$. Similarly we have that $(T\tilde{f}_n)_{n\in\mathbb{N}}$ converges in L^q. Let $\tilde{g}$ be its limit.
Now, we introduce $(h_n)_{n\in\mathbb{N}}$ such that $h_{2n} = f_{2n}$ and $h_{2n+1} = \tilde{f}_{2n+1}$. It is clear that $h_n \to f$ in $(L^p, ||.||_p)$ so it exists $h \in L^q$ such that $T(h_n) \to h$ in $(L^q, ||.||_q)$.
Thus, $T(h_{2n}) \to h$ but $T(h_{2n}) = T(f_{2n}) \to g$
and $T(h_{2n+1}) \to h$ but $T(h_{2n+1}) = T(\tilde{f}_{2n+1}) \to \tilde{g}$
Consequentially, $g = \tilde{g} = h$, so we founded an extension $\tilde{T}$ and we showed that

$$\forall f \in L^p, \tilde{T}(f) = \lim_{n\to+\infty} T(f_n)$$

Now we only have to show that $\tilde{T}$ is continuous and has the same norm as T.
Let $f \in L^p$, it exists $(f_n)_{n\in\mathbb{N}} \subset L^p \cap Et(\mu)$ such that $f_n \to f$ in $(L^p, ||.||_p)$.
Thus,

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}\ \forall n > N,\ \|f_n\|_p \leqslant \|f\|_p + \epsilon \text{ and } \|\tilde{T}(f) - T(f_n)\|_q \leqslant \epsilon$$

Finally we have that: $\forall n > N$

$$\begin{array}{rcl} \|\tilde{T}(f)\|_q & \leqslant & \|\tilde{T}(f) - T(f_n)\|_q + \|T(f_n)\| \\ & \leqslant & \epsilon + \|T\|_{p\to q} \times \|f_n\|_p \\ & \leqslant & \epsilon + \|T\|_{p\to q} \times (\|f\|_p + \epsilon) \\ & \leqslant & \|T\|_{p\to q} \times \|f\|_p + (\|T\|_{p\to q} + 1)\epsilon \end{array}$$

We do have as ϵ goes to 0 that $\|\tilde{T}(f)\|_q \leqslant \|T\|_{p\to q} \times \|f\|_p$ which concludes the proof of this proposition.

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<u>Proof :</u>

Now let's prove Riesz-Thorin theorem.
We will write $r^\star$ for the conjugate exponent of $r \in [1; \infty]$.
First, we assume that X and Y are complex spaces. Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be the step functions defined by:

$$f(s) = \sum_{k=1}^{m} a_k \mathbb{1}_{A_k}(s) \text{ and } g(t) = \sum_{l=1}^{n} b_l \mathbb{1}_{B_l}(t)$$

where $\forall (k,l) \in [\![1;m]\!] \times [\![1;n]\!]$, $a_k \neq 0$ and $b_l \neq 0$, $A_1, ..., A_m$ are pairwise disjoint, $B_1, ..., B_n$ are also pairwise disjoint, with strictly positive measure and such that $f \in L^p(\mu)$, $g \in L^{q^\star}(\nu)$ and $\|f\|_p = \|g\|_{q^\star} = 1$
Then we can define for $1 \leqslant p < \infty$:

$$F(s) = \begin{cases} |a_k|^p & \text{if } s \in A_k \\ 1 & \text{if } f(s) = 0 \end{cases} \qquad \text{and } u(s) = \begin{cases} e^{i \arg(a_k)} & \text{if } s \in A_k \\ 0 & \text{if } f(s) = 0 \end{cases}$$

We take $F \equiv 1$ and $u = f$ when $p = \infty$.
Similarly, for $1 \leqslant q^\star < \infty$, we define:

$$G(t) = \begin{cases} |b_l|^{q^\star} & \text{if } t \in B_l \\ 1 & \text{if } g(t) = 0 \end{cases} \qquad \text{and } v(t) = \begin{cases} e^{i \arg(b_l)} & \text{if } t \in B_l \\ 0 & \text{if } g(t) = 0 \end{cases}$$

and $G \equiv 1$ and $v = g$ when $q^\star = \infty$.
Thus, it is clear that we have:

$$f(s) = u(s)F(s)^{\frac{1}{p}} \text{ and } g(t) = v(t)G(t)^{\frac{1}{q^\star}} \qquad \forall s \in X \text{ and } \forall t \in Y.$$

By definition, F and G are strictly positive and

$$\begin{array}{rcl} \|f\|_p & = & \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} \\ & = & \left(\int_X |uF^{\frac{1}{p}}|^p d\mu\right)^{\frac{1}{p}} \\ & = & \left(\int_X |F| d\mu\right)^{\frac{1}{p}} \end{array}$$

But $\|f\|_p = 1$ and F is strictly positive, thus $\displaystyle\int_X F d\mu = 1$.
Similarly, $\displaystyle\int_Y G d\nu = 1$.
Moreover, $|u| \leqslant 1$ and $|v| \leqslant 1$ by definition.
Furthermore, we define:

$$\langle Tf, g\rangle = \int_Y (Tf) g \, d\nu = \int_Y T\left(uF^{\frac{1}{p}}\right) vG^{\frac{1}{q^\star}} d\nu$$

Now, for all $z \in \mathbb{C}$ we define:

$$\begin{cases} f_z(s) = u(s)\left(F(s)\right)^{\frac{1-z}{p_0}+\frac{z}{p_1}} & \forall s \in X \\ g_z(t) = v(t)\left(G(t)\right)^{\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}} & \forall t \in Y \end{cases}$$

and $\Phi(z) = \displaystyle\int_Y (Tf_z) g_z \, d\nu = \langle Tf_z, g_z\rangle$.
By definition:

$$\begin{cases} f_z = \displaystyle\sum_{k=1}^{m} e^{i\arg(a_k)} |a_k|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} \mathbb{1}_{A_k} \\ g_z = \displaystyle\sum_{l=1}^{n} e^{i\arg(b_l)} |b_l|^{q^\star\left(\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}\right)} \mathbb{1}_{B_l} \end{cases}$$

So we have:

$$\begin{aligned} \Phi(z) &= \langle Tf_z, g_z\rangle \\ &= \left\langle T\left(\sum_{k=1}^{m} e^{i\arg(a_k)} |a_k|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} \mathbb{1}_{A_k}\right), \sum_{l=1}^{n} e^{i\arg(b_l)} |b_l|^{q^\star\left(\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}\right)} \mathbb{1}_{B_l}\right\rangle \\ &= \left\langle \sum_{k=1}^{m} e^{i\arg(a_k)} |a_k|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} T\left(\mathbb{1}_{A_k}\right), \sum_{l=1}^{n} e^{i\arg(b_l)} |b_l|^{q^\star\left(\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}\right)} \mathbb{1}_{B_l}\right\rangle \\ &= \sum_{k=1}^{m} e^{i\arg(a_k)} |a_k|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} \sum_{l=1}^{n} e^{i\arg(b_l)} |b_l|^{q^\star\left(\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}\right)} \left\langle T\left(\mathbb{1}_{A_k}\right), \mathbb{1}_{B_l}\right\rangle \end{aligned}$$

Thus Φ is holomorphic on $\mathbb{C}$ as a finite sum of holomorphic functions
(since $z \in \mathbb{C} \longmapsto e^{z\ln(a)} \in \mathbb{C}$ is holomorphic for all $a > 0$.) Moreover $\Phi(z)$ is bounded by a quantity independant of z for all z such that $0 \leqslant Re(z) \leqslant 1$. Indeed we thus have:

$$\begin{aligned} |\Phi(z)| &\leqslant \sum_{k=1}^{m}\sum_{l=1}^{n} \left|e^{i(\arg(a_k)+\arg(b_l))}\right| \left|e^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)\ln|a_k|}\right| \left|e^{q^\star\left(\frac{1-z}{q_0^\star}+\frac{z}{q_1^\star}\right)\ln|a_k|}\right| \left|\left\langle T\left(\mathbb{1}_{A_k}\right), \mathbb{1}_{B_l}\right\rangle\right| \\ &\leqslant \sum_{k=1}^{m}\sum_{l=1}^{n} \left|e^{p\left(\frac{1-Re(z)}{p_0}+\frac{Re(z)}{p_1}\right)\ln|a_k|}\right| \left|e^{q^\star\left(\frac{1-Re(z)}{q_0^\star}+\frac{Re(z)}{q_1^\star}\right)\ln|a_k|}\right| \left|\left\langle T\left(\mathbb{1}_{A_k}\right), \mathbb{1}_{B_l}\right\rangle\right| \\ &\leqslant \sum_{k=1}^{m}\sum_{l=1}^{n} \left|\max\left(|a_k|^{p\left(\frac{1}{p_0}+\frac{1}{p_1}\right)}; 1\right)\right| \left|\max\left(|b_l|^{q^\star\left(\frac{1}{q_0^\star}+\frac{1}{q_1^\star}\right)}; 1\right)\right| \left|\left\langle T\left(\mathbb{1}_{A_k}\right), \mathbb{1}_{B_l}\right\rangle\right| \end{aligned}$$

Therefore, we can apply the three lines theorem to Φ. Before that, we notice that:

$$\begin{array}{rcl} |f_{iy}| & = & \left|\displaystyle\sum_{k=1}^{m} e^{i\arg(a_k)}|a_k|^{p\left(\frac{1-iy}{p_0}+\frac{iy}{p_1}\right)}\mathbb{1}_{A_k}\right| \\ & \leqslant & \displaystyle\sum_{k=1}^{m}|a_k|^{\frac{p}{p_0}}\mathbb{1}_{A_k} \\ & \leqslant & |F|^{\frac{1}{p_0}} \end{array}$$

And, similarly $|g_{iy}| \leqslant |G|^{\frac{1}{q_0^\star}}$.
Thus, $\|f_{iy}\|_{p_0} \leqslant 1$ and $\|g_{iy}\|_{q_0^\star} \leqslant 1$ because :

$$\int_X |f_{iy}|^{p_0} d\mu \leqslant \int_X |F|^{\frac{p_0}{p_0}} d\mu \leqslant 1$$

(and the same thing goes for g.)
Therefore, by Hölder inequality we have that :

$$\begin{array}{rcl} |\Phi(iy)| & \leqslant & \|T(f_{iy})\|_{q_0}\ \|g_{iy}\|_{q_0^\star} \\ & \leqslant & \|T\|_{p_0\to q_0}\ \|f_{iy}\|_{p_0}\ \|g_{iy}\|_{q_0^\star} \\ & \leqslant & \|T\|_{p_0\to q_0} \end{array}$$

Likewise we find that $|f_{1+iy}| \leqslant |F|^{\frac{1}{p_1}}$ and $|g_{1+iy}| \leqslant |G|^{\frac{1}{q_1^\star}}$.
Thus $\|f_{1-iy}\|_{p_1} \leqslant 1$ and $\|g_{1+iy}\|_{q_1^\star} \leqslant 1$ and therefore, by Hölder inequality we have that:

$$|\Phi(1+iy)| \leqslant \|T\|_{p_1\to q_1}$$

Let θ be in $]0;1[$, by the three lines theorem we thus have that:
$\forall y \in \mathbb{R}$, $|\Phi(\theta+iy)| \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}$. In particular $|\Phi(\theta)| \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}$.
But $|\Phi(\theta)| = |\langle Tf, g\rangle|$. Indeed,

$$\forall s \in X,\ f_\theta(s) = u(s)\left(F(s)^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}\right) = u(s)\left(F(s)^{\frac{1}{p}}\right) = f(s)$$

Similarly $g_\theta = g$ because if $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ then $\frac{1}{q^\star} = \frac{1-\theta}{q_0^\star} + \frac{\theta}{q_1^\star}$. Indeed,

$$\begin{array}{rcl} \dfrac{1}{q} + \left(\dfrac{1-\theta}{q_0^\star} + \dfrac{\theta}{q_1^\star}\right) & = & \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1} + \dfrac{1-\theta}{q_0^\star} + \dfrac{\theta}{q_1^\star} \\ & = & (1-\theta)\left(\dfrac{1}{q_0^\star} + \dfrac{1}{q_0}\right) + \theta\left(\dfrac{1}{q_1^\star} + \dfrac{1}{q_1}\right) \\ & = & (1-\theta) + \theta \\ & = & 1 \end{array}$$

So by definition of the conjugate exponent: $\frac{1}{q^\star} = \frac{1-\theta}{q_0^\star} + \frac{\theta}{q_1^\star}$ and we can do the same computation that we did for f_θ but for g_θ. So we do have that $|\Phi(\theta)| = |\langle T(f_\theta), g_\theta\rangle| = |\langle Tf, g\rangle|$
So we proved that for every step function g whose norm in $L^{q^\star}$ is less than or equal to 1:

$$|\langle Tf, g\rangle| \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}$$

We can apply the previous lemma to Tf and we thus have that for all step functions f of $L^p(\mu)$ such that $\|f\|_p = 1$

$$\|Tf\|_q \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}$$

Finally for any step function f in $L^p(\mu)$ we define $\widetilde{f}=\frac{f}{\|f\|}$ so that $\|\widetilde{f}\| = 1$ and :

$$\|Tf\|_q = \|T(\widetilde{f}\ \|f\|)\|_q = \|f\|_p\ \|T(\widetilde{f})\|_q$$

so

$$\|Tf\| \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}\ \|f\|_p$$

Ultimately by applying the previous proposition to T we have:

$$\|T\|_{p\to q} \leqslant \|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta}$$

For real spaces, we only have to define $\widetilde{T}$ by $\widetilde{T}\,(f+ig) = (Tf) + i(Tg)$ to bring ourselves back in the complex case. Therefore we can just notice that for all $1 \leqslant r, s \leqslant \infty$, we have

$$\begin{aligned} \|T\|_{r\to s} \leqslant \|\widetilde{T}\|_{r\to s} \leqslant 2\|T\|_{r\to s} \quad &\Longrightarrow \quad \|T\|_{r\to s} \leqslant \|\widetilde{T}\|_{p_0\to q_0}^{1-\theta}\ \|\widetilde{T}\|_{p_1\to q_1}^{\theta} \\ &\Longrightarrow \quad \|T\|_{r\to s} \leqslant 2^{1-\theta+\theta}\|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta} \\ &\Longrightarrow \quad \|T\|_{r\to s} \leqslant 2\|T\|_{p_0\to q_0}^{1-\theta}\ \|T\|_{p_1\to q_1}^{\theta} \end{aligned}$$

which concludes the proof. ■

IV Some applications of the theorem

IV.1 The Hausdorff-Young theorem

Definition : Fourier coefficients

Let $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$.
The n-th Fourier coefficient of f, named $\widehat{f}(n)$, is defined by :

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\mathrm{d}t$$

We then can define the operator

$$\widehat{\cdot} : \left| \begin{array}{ccc} L^1(\mathbb{T}) & \longrightarrow & \ell^\infty(\mathbb{Z}) \\ f & \longmapsto & \left(\widehat{f}(n)\right)_{n\in\mathbb{Z}} \end{array} \right.$$

Theorem : Hausdorff-Young theorem (applied to Fourier series)

For every $p \in [1\,;2]$, and for every $q \in [2\,;+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$, the operator $\widehat{\cdot}$ is a linear continuous map from $L^p(\mathbb{T})$ to $\ell^q(\mathbb{Z})$ with a norm $\|\widehat{\cdot}\|_{p\to q}$ lower than 1.

Proof :

To begin with, it is clear that this operator is linear.
Indeed, let f and g be two maps from $\mathbb{T}$ to $\mathbb{C}$ and $n \in \mathbb{Z}$:

$$\begin{aligned}\widehat{(f+g)}(n) &= \frac{1}{2\pi}\int_0^{2\pi}(f(t)+g(t))e^{-int}\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\mathrm{d}t + \frac{1}{2\pi}\int_0^{2\pi} g(t))e^{-int}\mathrm{d}t \\ &= \widehat{f}(n)+\widehat{g}(n)\end{aligned}$$

Furthermore, $\widehat{\cdot}$ is continuous from $L^1(\mathbb{T})$ to $\ell^\infty(\mathbb{Z})$.
Indeed, let us consider $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$.

$$\begin{aligned}\left|\widehat{f}(n)\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\mathrm{d}t\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}|f(t)|\,\mathrm{d}t \\ &\leq \|f\|_{L^1(\mathbb{T})} < +\infty\end{aligned}$$

Moreover, $\widehat{\cdot}$ is continuous from $L^2(\mathbb{T})$ to $\ell^2(\mathbb{Z})$.
This comes from the fact that $(e_n)_{n\in\mathbb{Z}}$ is a Hilbertian base of $L^2(\mathbb{T})$ (where for every $n \in \mathbb{Z}$, for every $\theta \in \mathbb{R}, e_n(\theta) = e^{in\theta}$).
By the Parseval theorem, we then have : $\displaystyle\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right|^2 = \|f\|_{L^2(\mathbb{T})} < +\infty$.
So $\widehat{\cdot}$ is a continuous linear map from $L^1(\mathbb{T})$ to $\ell^\infty(\mathbb{Z})$ and from $L^2(\mathbb{T})$ to $\ell^2(\mathbb{Z})$.
We can apply the Riesz-Thorin theorem with the measured spaces $\left(\mathbb{T}, \mathcal{B}(\mathbb{T}), \dfrac{dt}{2\pi}\right)$ and $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \nu)$ where ν is the counting measure.
For every $p \in [1\,;2]$, for every $q \in [2\,;+\infty]$ such that $\dfrac{1}{p}+\dfrac{1}{q} = 1$, $\widehat{\cdot}$ is a continuous linear map from $L^p(\mathbb{T})$ to $\ell^q(\mathbb{Z})$.
Furthermore, thanks to our earlier computations, we have that $\|\widehat{\cdot}\|_{1\to+\infty} \leq 1$ and $\|\widehat{\cdot}\|_{2\to 2} = 1$.
We finally can conclude, thanks again to the Riesz-Thorin theorem, that :

$$\|\widehat{\cdot}\|_{p\to q} \leq 1$$

Definition : Fourier transform

Let $f \in L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$.
We define the Fourier transform of f in ξ, named $\mathcal{F}f(\xi)$ as :

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t)e^{-2i\pi t\xi}\mathrm{d}t$$

We can also define a Fourier transform for elements of $L^2(\mathbb{R})$, which will be used later to apply the Riesz-Thorin theorem to the operator $\mathcal{F}$.
First, we need a definition of the convolution.

Definition : Convolution of two functions

Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $p \in [1\,;+\infty]$.
We define the convolution of f and g as following :

$$f * g : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_{\mathbb{R}} f(t)g(x-t)\mathrm{d}t \end{array} \right.$$

This function is defined for almost every $x \in \mathbb{R}$, and we also have that $f * g \in L^1(\mathbb{R})$.

To extend $\mathcal{F}$ to functions in $L^2(\mathbb{R})$, we also need an approximation of the convolution unity, defined for $s > 0$ as such :

$$\gamma_s : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{\sqrt{s}} e^{-\dfrac{\pi x^2}{s}} \end{array} \right.$$

We are now ready to define the Fourier transform for L^2 functions.

Theorem : Fourier-Plancherel transform

Let $f \in L^2(\mathbb{R})$.
Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, let us consider $(f_n) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that it tends to f in norm $\|\cdot\|_2$
Then $(\mathcal{F}f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and we have :

$$\mathcal{F}f := \lim_{n\to+\infty} \mathcal{F}f_n$$

which we call the Fourier-Plancherel transform of f.

Proof :

$\mathcal{F}$ is a isometry on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$:
Let $f \in L^1(\mathbb{R})$ such that $\mathcal{F}f \in L^1(\mathbb{R})$.

$$\begin{aligned} \|\mathcal{F}f\|_{L^2} &= \int_{\mathbb{R}} \mathcal{F}f\overline{(\mathcal{F}f)} \\ &= \int_{\mathbb{R}} \mathcal{F}f(\overline{\mathcal{F}}\overline{f}) \\ &= \int_{\mathbb{R}} \overline{\mathcal{F}}(\mathcal{F}f)\overline{f} \\ &= \int_{\mathbb{R}} f\overline{f} \\ &= \|f\|_{L^2} \end{aligned}$$

If we consider $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then we do the same computations with $f * \gamma_{\frac{1}{n}}, n \in \mathbb{N}^*$, and we get the same result.
Indeed :

$$\left\|\mathcal{F}(f * \gamma_{\frac{1}{n}})\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |(\mathcal{F}f)(t)|\, e^{-2\pi\frac{t^2}{n}}\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}} |\mathcal{F}f|^2 \text{ thanks to the Beppo Levi theorem}$$

The extension is independent of the choice made for (f_n)
Indeed, let us consider (f_n) and (F_n) two sequences of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that converge to f in

norm $\|\cdot\|_{L^2(\mathbb{R})}$.
Let us pose (Φ_n) such that : $\forall n \in \mathbb{N}, \Phi_{2n} = f_n, \Phi_{2n+1} = F_n$.
(Φ_n) is a Cauchy sequence, which means that (f_n) and (F_n) have the same limit.

Conclusion :

- $\mathcal{F} \colon L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is linear ;
- $\mathcal{F} \colon L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is an isometry :

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})} = \left\|\mathcal{F}\left(\lim_{n\to+\infty} f_n\right)\right\|_{L^2(\mathbb{R})} = \left\|\lim_{n\to+\infty} \mathcal{F}f_n\right\|_{L^2(\mathbb{R})} = \lim_{n\to+\infty} \|\mathcal{F}f_n\|_{L^2(\mathbb{R})} = \lim_{n\to+\infty} \|f_n\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

- $\operatorname{Im}\mathcal{F}$ is a closed and dense set ;

We have a linear map $\mathcal{F}$ uniformly continuous on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, a dense subset of $L^2(\mathbb{R})$.
Then there exists a unique continuous linear map which extends $\mathcal{F}$ on $L^2(\mathbb{R})$.

■

Now that $\mathcal{F}$ is well defined on L^2, let's apply the Riesz-Thorin theorem to this operator.

Theorem : Hausdorff-Young theorem

For every $p \in [1\,;2]$, and for every $q \in [2\,;+\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the operator $\mathcal{F}$ is a linear continuous map from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ with a norm $\|\mathcal{F}\|_{p\to q}$ lower than 1.

Proof :

As shown earlier, we have $\|\mathcal{F}\|_{L^2(\mathbb{R})} = 1$.
Furthermore, let $f \in L^1(\mathbb{R})$.

$$\begin{aligned}
\forall x \in \mathbb{R}, |(\mathcal{F}f)(x)| &= \left|\int_{\mathbb{R}} f(t)e^{-2i\pi xt}\mathrm{d}t\right| \\
&\leq \int_{\mathbb{R}} \left|f(t)e^{-2i\pi xt}\right| \mathrm{d}t \\
&\leq \int_{\mathbb{R}} |f(t)|\, \mathrm{d}t \\
&\leq \|f\|_{L^1(\mathbb{R})}
\end{aligned}$$

We then conclude that $\|\mathcal{F}\|_{L^\infty} \leq 1$.
So, $\mathcal{F}$ is a continuous linear operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with an operator norm equal to 1, and is also a continuous linear operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ with an operator norm inferior to 1.
Thanks to the Riesz-Thorin theorem, we have that for every $\theta \in [0\,;1]$, the operator $\mathcal{F}$ is continuous from $L^{p_\theta}(\mathbb{R})$ to $L^{q_\theta}(\mathbb{R})$, where we have $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{1} + \dfrac{\theta}{2} = 1 - \dfrac{\theta}{2}$ and $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{\infty} + \dfrac{\theta}{2} = \dfrac{\theta}{2}$, so $q_\theta = p_\theta^*$.
As θ varies in $[0\,;1]$, p_θ varies in $[1\,;2]$, and so q_θ varies in $[2\,;+\infty]$.
We then conclude that for every $p \in [1\,;2]$ and for every $q \in [2\,;+\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we

have $\mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

■

IV.2 Convolution

Proposition :

Let $1 \leqslant \alpha < +\infty$ and f a fixed function of $L^\alpha(\mathbb{R}^n) = L^\alpha$. Let T be the operator of convolution with f i.e $Tg = f * g$. Let α' be the conjugate exponent. Then T is a linear continuous application from L^β to L^γ where β and γ are defined by: $\forall\ 0 < \theta < 1$

$$\frac{1}{\beta} = \frac{1-\theta}{\alpha'} + \theta, \ \frac{1}{\gamma} = \frac{1-\theta}{\infty} + \frac{\theta}{\alpha}$$

Moreover $\|Tg\|_\gamma \leqslant \|f\|_\alpha \cdot \|g\|_\beta$.

Proof :

First, it is clear that the operator of convolution is linear when it is well defined. We are going to prove that T is continuous from $L^{\alpha'}$ to L^∞ and $\|T\|_{\alpha' \to \infty} \leqslant \|f\|_\alpha$.
By Hölder inequality, we have: $\forall x \in \mathbb{R}^n$,

$$|(f * g)(x)| \leqslant \int |f(x-y)| \cdot |g(y)| dy \leqslant \Big(\int |f(x-y)|^\alpha dy\Big)^{\frac{1}{\alpha}} \Big(\int |g(y)|^{\alpha'} dy\Big)^{\frac{1}{\alpha'}}$$

i.e

$$\|f * g\|_\infty \leqslant \|f\|_\alpha \cdot \|g\|_{\alpha'}$$

And the case $\alpha = 1$, $\alpha' = \infty$ is immediate.
Now, we will show that T is continuous from L^1 to L^α and $\|T\|_{1 \to \alpha} \leqslant \|f\|_\alpha$
Since $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, we have by Hölder inequality:

$$\begin{aligned}
\int |f(x-y)|\,|g(y)|\,dy &= \int |f(x-y)|\,|g(y)|^{\frac{1}{\alpha}}\,|g(y)|^{\frac{1}{\alpha'}}\,dy \\
&\leqslant \Big(\int |f(x-y)|^\alpha\,|g(y)|\,dy\Big)^{\frac{1}{\alpha}} \cdot \Big(\int |g(y)|\,dy\Big)^{\frac{1}{\alpha'}}
\end{aligned}$$

Then, by Fubini-Tonelli theorem,

$$\begin{aligned}
\int \Big[\int |f(x-y)|\,|g(y)|\,dy\Big]^\alpha dx &\leqslant \int \Big[\Big(\int |f(x-y)|^\alpha\,|g(y)|\,dy\Big) \cdot \Big(\int |g(y)|\,dy\Big)^{\frac{\alpha}{\alpha'}}\Big] dx \\
&\leqslant \Big(\int |g(y)|\,dy\Big)^{\frac{\alpha}{\alpha'}} \cdot \int\int |f(x-y)|^\alpha\,|g(y)|\,dx\,dy \\
&\leqslant \Big(\int |g(y)|\,dy\Big)^{\frac{\alpha}{\alpha'}} \cdot \int |f(x-y)|^\alpha\,dx \cdot \int |g(y)|\,dy \\
&\leqslant \Big(\int |g(y)|\,dy\Big)^{\frac{\alpha}{\alpha'}} \cdot \int |f(x)|^\alpha\,dx \cdot \int |g(y)|\,dy \\
&\leqslant \Big(\int |g(y)|\,dy\Big)^{\frac{\alpha}{\alpha'}+1} \cdot \int |f(x)|^\alpha\,dx
\end{aligned}$$

Thus, since $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ we have:

$$\|f * g\|_\alpha \leqslant \|g\|_1 \cdot \|f\|_\alpha$$

By applying Riesz-Thorin theorem to $T : L^{\alpha'} \to L^\infty$ and $T : L^1 \to L^\alpha$ we have what we wanted.

$\blacksquare$

IV.3 Clarkson's inequalities

This other application of the Riesz-Thorin theorem is a result of the L^p-theory named after James Andrew Clarkson.
Its goal is to be a substitution to the parallelogram law in L^p-spaces.

Lemma :

Let $(a, b) \in \mathbb{C}^2$.

If $p \in [1\,;2]$, then : $\left(|a+b|^{p^*} + |a-b|^{p^*}\right)^{\frac{1}{p^*}} \leq 2^{\frac{1}{p^*}} \left(|a|^p + |b|^p\right)^{\frac{1}{p}}$

If $p \geq 2$, then : $\left(|a+b|^p + |a-b|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p^*}} \left(|a|^p + |b|^p\right)^{\frac{1}{p}}$

Remark :

To prove this lemma, we will identify $\mathbb{C}^2$ as $\ell_p^2 := L^p(\{1,2\}, \nu)$, where ν is the counting measure.

In this context, we then have : $\forall (a,b) \in \mathbb{C}^2, \|(a,b)\|_{\ell_p} = \left(|a|^p + |b|^p\right)^{\frac{1}{p}}$

Proof :

Let us pose the operator

$$T : \left| \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (a,b) & \longmapsto & (a+b, a-b) \end{array} \right.$$

It is clear that T is a linear operator.

First case : $p \in [1\,;2]$

On the one hand, we have :

$$\begin{aligned} \|T\|_{\ell_2 \to \ell_2} &= \sup_{|a|^2+|b|^2=1} \sqrt{|a+b|^2 + |a-b|^2} \\ &= \sup_{|a|^2+|b|^2=1} \sqrt{2(|a^2| + |b|^2)} \text{ by the parallelogram identity} \\ &= \sqrt{2} \end{aligned}$$

On the other hand, we have :

$$\|T\|_{\ell_1 \to \ell_\infty} = \sup_{|a|+|b|=1} \max(|a+b|\,, |a-b|) \leq 1$$

and since $\|T(0,1)\|_{\ell_\infty} = \max(|1|\,, |-1|) = 1 = \|(0,1)\|_{\ell_1}$, we have the equality.

We showed that T is continuous from ℓ_2 to ℓ_2 and from ℓ_1 to ℓ_∞.
Thanks to the Riesz-Thorin theorem, we have :

$$\forall \theta \in [0\,;1], T \in \mathcal{L}(\ell_{p_\theta}^2, \ell_{q_\theta}^2)$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{2} + \dfrac{\theta}{1} = \dfrac{1-\theta}{2} + \theta = \dfrac{1+\theta}{2}$ and $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{2} + \dfrac{\theta}{\infty} = \dfrac{1-\theta}{2}$ (so p_θ and q_θ are conjugate.)
We also have that $\frac{1}{p_\theta} \in [\frac{1}{2}\,;1]$, so $p_\theta \in [1\,;2]$.
Furthermore, we have

$$\|T\|_{\ell_{p_\theta}\to\ell_{q_\theta}} \leq \|T\|^{1-\theta}_{\ell_2\to\ell_2}\,\|T\|^{\theta}_{\ell_1\to\ell_\infty}$$

that is to say

$$\|T\|_{\ell_{p_\theta}\to\ell_{q_\theta}} \leq 2^{\frac{1-\theta}{2}} = 2^{\frac{1}{q_\theta}}$$

To conclude, since $p_\theta \in [1\,;2]$, we can choose θ so that $p = p_\theta$; we then write the following estimation for all $(a,b) \in \mathbb{C}^2$:

$$\|T(a,b)\|_{\ell_{p^*}} \leq \|T\|_{\ell_p\to\ell_{p^*}}\,\|(a,b)\|_{\ell_p}$$
$$\left(|a+b|^{p^*} + |a-b|^{p^*}\right)^{\frac{1}{p^*}} \leq 2^{\frac{1}{p^*}}\left(|a|^p + |b|^p\right)^{\frac{1}{p}}$$

Last case : $p \geq 2$
As shown earlier, we have $\|T\|_{\ell_2\to\ell_2} = \sqrt{2}$.
Furthermore, we have :

$$\begin{aligned}
\|T\|_{\ell_\infty\to\ell_\infty} &= \sup_{\max(|a|,|b|)=1} \max(|a+b|\,,|a-b|) \\
&\leq \sup_{\max(|a|,|b|)=1} |a|+|b| \text{ by the triangle inequality} \\
&\leq 2
\end{aligned}$$

Since $\|T(1,1)\|_{\ell_\infty} = \max(2,0) = 2 = 2\,\|(1,1)\|_{\ell_\infty}$, we have the equality.
Thanks again to the Riesz-Thorin theorem, we have :

$$\forall\theta \in [0\,;1], T \in \mathcal{L}(\ell_{p_\theta}, \ell_{q_\theta})$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{2} + \dfrac{\theta}{\infty} = \dfrac{1-\theta}{2}$ and $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{2} + \dfrac{\theta}{\infty} = \dfrac{1-\theta}{2}$.
Furthermore, we have

$$\|T\|_{\ell_{p_\theta}\to\ell_{q_\theta}} \leq 2^{\frac{1-\theta}{2}}2^{\theta} = 2^{\frac{1+\theta}{2}}$$

We also remark that $\frac{1}{p_\theta} \in [0\,;\frac{1}{2}]$, so $p_\theta \in [2\,;+\infty[$.
Let us then choose θ so that $p = p_\theta$ (then $p^* = \dfrac{1+\theta}{2}$) ; thanks to the same computations as earlier, we find for all $(a,b) \in \mathbb{C}^2$:

$$\|T(a,b)\|_{\ell_p} \leq \|T\|_{\ell_{p_\theta}\to\ell_{q_\theta}}\,\|(a,b)\|_{\ell_p}$$

that is to say

$$\left(|a+b|^p + |a-b|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p^*}}\left(|a|^p + |b|^p\right)^{\frac{1}{p}}$$

■

We are now ready to prove the inequalities.

Theorem : Clarkson's inequalities

Let $(X, \mathcal{A}, \mu)$ be a measured space.

- If $p \in [1\,;2]$, then :
$\forall f, g \in L^p(X), \left(\|f+g\|_p^{p^*} + \|f-g\|_p^{p^*}\right)^{\frac{1}{p^*}} \leq 2^{\frac{1}{p^*}} \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}}$
- If $p \geq 2$, then :
$\forall f, g \in L^p(X), \left(\|f+g\|_p^p + \|f-g\|_p^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p^*}} \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}}$

Proof :

Let f, g be in $L^p(X)$.

First case : $p \in [1\,;2]$

$$\begin{aligned}
\|f+g\|_p^{p^*} + \|f-g\|_p^{p^*} &= \left(\int_X |f(t)+g(t)|^p \,\mathrm{d}\mu\right)^{\frac{p^*}{p}} + \left(\int_X |f(t)-g(t)|^p \,\mathrm{d}\mu\right)^{\frac{p^*}{p}} \\
&= \left(\int_X |f(t)+g(t)|^{p^* \frac{p}{p^*}} \,\mathrm{d}\mu\right)^{\frac{p^*}{p}} + \left(\int_X |f(t)-g(t)|^{p^* \frac{p}{p^*}} \,\mathrm{d}\mu\right)^{\frac{p^*}{p}} \\
&= \left\| \, |f+g|^{p^*} \right\|_{\frac{p}{p^*}} + \left\| \, |f-g|^{p^*} \right\|_{\frac{p}{p^*}}
\end{aligned}$$

But $p \in [1\,;2]$, so its conjugate exponent p^* is greater than 2, and we have $\frac{p}{p^*} < 1$, so thanks to the reversed Minkowski inequality :

$$\begin{aligned}
\|f+g\|_p^{p^*} + \|f-g\|_p^{p^*} &\leq \left\| \, |f+g|^{p^*} + |f-g|^{p^*} \right\|_{\frac{p}{p^*}} \\
&\leq \left(\int_X \left(|f(t)+g(t)|^{p^*} + |f(t)-g(t)|^{p^*}\right)^{\frac{p}{p^*}} \mathrm{d}\mu\right)^{\frac{p^*}{p}} \\
&\leq \left(\int_X 2^{\frac{p}{p^*}} \left(|f(t)|^p + |g(t)|^p\right) \mathrm{d}\mu\right)^{\frac{p^*}{p}} \quad \text{thanks to the lemma proved earlier} \\
&\leq 2 \left(\int_X |f(t)|^p \,\mathrm{d}\mu + \int_X |g(t)|^p \,\mathrm{d}\mu\right)^{\frac{p^*}{p}} \\
&\leq 2 \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{p^*}{p}}
\end{aligned}$$

So we conclude that :

$$\left(\|f+g\|_p^{p^*} + \|f-g\|_p^{p^*}\right)^{\frac{1}{p^*}} \leq 2^{\frac{1}{p^*}} \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}}$$

Last case : $p \geq 2$

$$\begin{aligned}
\|f+g\|_p^p + \|f-g\|_p^p &= \int_X |f(t)+g(t)|^p \,\mathrm{d}\mu + \int_X |f(t)-g(t)|^p \,\mathrm{d}\mu \\
&= \int_X \left(|f(t)+g(t)|^p + |f(t)-g(t)|^p\right) \mathrm{d}\mu \\
&\leq 2^{\frac{p}{p^*}} \int_X \left(|f(t)|^p + |g(t)|^p\right) \mathrm{d}\mu \text{ thanks to the lemma proved earlier} \\
&\leq 2^{\frac{p}{p^*}} \left(\|f\|_p^p + \|g\|_p^p\right)
\end{aligned}$$

So we conclude that :

$$\left(\|f+g\|_p^p + \|f-g\|_p^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p^*}} \left(\|f\|_p^p + \|g\|_p^p\right)^{\frac{1}{p}}$$

■

V Interpolation of Banach spaces

The Riesz-Thorin theorem is a result of complex interpolation between L^p spaces. One could wonder if more general results exist. That is what interpolation spaces or interpolation theory is: a construction of intermediate spaces first, and interpolation spaces then, between two Banach spaces.
This theory was developed during the sixties by mathematicians such has A. Calderon.

Such a construction re uses some of the arguments we presented during the proof of the Riesz-Thorin theorem (more specifically the maximum principle and the three lines theorem).

In this document we won't give explicit proofs but we will confine ourselves to the presentation of the most significant results as to the construction A. Calderon wrote in 1964 in his paper: "Intermediate spaces and interpolation, the complex method".

First, we need to introduce some definitions.

Definition :

A couple of Banach spaces (X, Y) is said to be an interpolation couple if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$.

Lemma :

In this case, the intersection $X \cap Y$ is a linear subspace of $\mathcal{V}$, and it is a Banach space under the norm:

$$\|v\|_{X \cap Y} = \max\{\|v\|_X, \|v\|_Y\}$$

Moreover, the sum $X + Y := \{x + y : x \in X, y \in Y\}$ is a linear subspace of $\mathcal{V}$ and endowed with the following norm it is also a Banach space.

$$\|v\|_{X+Y} = \inf_{\substack{x \in X, y \in Y \\ x+y=v}} \|x\|_X + \|y\|_Y$$

Finally, $\|x\|_X \leqslant \|x\|_{X+Y}$ and $\|y\|_Y \leqslant \|x\|_{X+Y}$ for all $x \in X, y \in Y$, so that both X and Y are continuously embedded in $X + Y$.

Definition :

If (X, Y) is an interpolation couple, an intermediate space is any Banach space E such that:

$$X \cap Y \subset E \subset X + Y$$

An interpolation space between X and Y is any intermediate space such that for every $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ i.e for every linear operator $T : X + Y \to X + Y$ whose restriction to X belongs to $\mathcal{L}(X)$ and whose restriction to Y belongs to $\mathcal{L}(Y)$, the restriction of T to E belongs to $\mathcal{L}(E)$.

Lemma :

Let (X, Y) be an interpolation couple, and let E be an interpolation space between X and Y. Then there exists $C > 0$ such that for each linear operator $T : X + Y \to X + Y$ such that $T_{|X} \in \mathcal{L}(X), T_{|Y} \in \mathcal{L}(Y)$ we have

$$\|T\|_{\mathcal{L}(E)} \leqslant C \max\{\|T\|_{\mathcal{L}(X)}, \|T\|_{\mathcal{L}(Y)}\}.$$

Now that we know what an interpolation space is, we can begin to give a method to construct some of them.

Definition :

Let S be the strip $\{z = x+iy : 0 \leqslant x \leqslant 1\}$ and $\mathcal{F}(X, Y)$ the space of all functions $f : S \to X+Y$ such that

(i) f is holomorphic on $\mathring{S}$ and continuous and bounded up to its boundary, with values in $X + Y$

(ii) $t \mapsto f(it) \in \mathbf{C}_b(\mathbb{R}; X)$, $t \mapsto f(1 + it) \in \mathbf{C}_b(\mathbb{R}; X)$, and

$$\|f\|_{\mathcal{F}(X,Y)} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_Y\} < \infty$$

Moreover $\mathcal{F}_0(X, Y)$ is the subspace of $\mathcal{F}(X, Y)$ consisting of the functions $f : S \to X + Y$ such that

$$\lim_{|t|\to\infty} \|f(it)\|_X = 0, \quad \lim_{|t|\to\infty} \|f(1 + it)\|_Y = 0$$

Proposition :

$\mathcal{F}(X, Y)$ and $\mathcal{F}_0(X, Y)$ are Banach spaces, continuously embedded in $\mathbf{C}_b(S, X + Y)$.

Remark :

The proof of this proposition lays on the maximum principle and the completeness of $X + Y$.

Definition :

For every $\theta \in [0;1]$ the set

$$[X,Y]_\theta := \{f(\theta) : \ f \in \mathcal{F}(X,Y)\}, \text{ with the following norm}$$
$$\|a\|_{[X,Y]_\theta} := \inf_{f \in \mathcal{F}(X,Y), f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)}$$

is a Banach space.

Proposition :

We have some immediate consequences:

(i) For every $\theta \in (0,1)$,
$$[X,Y]_\theta = [X,Y]_{1-\theta}$$

(ii) If $X = Y$ then $[X,Y]_\theta = X$ with identical norms

(iii) For every $t \in \mathbb{R}$ and for every $f \in \mathcal{F}(X,Y)$, then $f(\theta + it) \in [X,Y]_\theta$ for each $\theta \in (0,1)$, and $\|f(\theta+it)\|_{[X,Y]_\theta} = \|f(\theta)\|_{[X,Y]_\theta}$

Remark :

We get an equivalent definition of $[X,Y]_\theta$ replacing the space $\mathcal{F}(X,Y)$ by the space $\mathcal{F}_0(X,Y)$ since
$$\inf_{f \in \mathcal{F}(X,Y), f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)} = \inf_{f \in \mathcal{F}_0(X,Y), f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)}$$

We now arrive to two theorems that can be seen as a generalization of the Riesz-Thorin theorem. First, let's give a proposition regarding interpolation of spaces.

Proposition :

Let $0 < \theta < 1$. Then,
$$X \cap Y \subset [X,Y]_\theta \subset X + Y.$$
i.e $[X,Y]_\theta$ is an intermediate space for the interpolation couple (X,Y).

Theorem :

Let $(X_1,Y_1),(X_2,Y_2)$ be complex interpolation couples. If a linear operator T belongs to $\mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$, then the restriction of T to $[X_1,Y_1]_\theta$ belongs to $\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)$ for every $\theta \in (0,1)$. Moreover,

$$\|T\|_{\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \leqslant (\|T\|_{\mathcal{L}[X_1,X_2]})^{1-\theta}(\|T\|_{\mathcal{L}[Y_1,Y_2]})^{\theta}$$

Theorem :

Let $(X_1, Y_1), (X_2, Y_2)$ be complex interpolation couples.
For every $z \in S$ let $T_z \in \mathcal{L}(X_1 \cap Y_1, X_2 + Y_2)$ be such that $z \mapsto T_z x$ is holomorphic in S and continuous and bounded in S for every $x \in X_1 \cap Y_1$, with values in $X_2 + Y_2$.
Moreover let us assume that $t \mapsto T_{it}x \in \mathbf{C}_b(\mathbb{R}; \mathcal{L}(X_1, X_2))$ and $t \mapsto T_{1+it}x \in \mathbf{C}_b(\mathbb{R}; \mathcal{L}(Y_1, Y_2))$.
Then, setting

$$M_0 := \sup_{t\in\mathbb{R}} \|T_{it}\|_{\mathcal{L}(X_1,X_2)}, \ M_1 := \sup_{t\in\mathbb{R}} \|T_{1+it}\|_{\mathcal{L}(Y_1,Y_2)}$$

for every $\theta \in (0, 1)$ we have

$$\|T_\theta x\|_{[X_2,Y_2]_\theta} \leqslant M_0^{1-\theta} M_1^\theta \|x\|_{[X_1,Y_1]_\theta}$$

so that T_θ has an unique extension belonging to $\mathcal{L}([X_1, Y_1]_\theta, [X_2, Y_2]_\theta)$ (which we still call T_θ) satisfying

$$\|T_\theta\|_{\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \leqslant M_0^{1-\theta} M_1^\theta$$

Corollary :

For every $\theta \in (0, 1)$ we have

$$\|y\|_{[X,Y]_\theta} \leqslant \|y\|_X^{1-\theta} \|y\|_Y^\theta, \ \forall y \in X \cap Y.$$

To finish this section, we will give an example of interpolation spaces.

Example :

Let (Ω, μ) be a measure space with σ-finite measure, and let $1 \leqslant p_0, p_1 \leqslant \infty$, $0 < \theta < 1$. Then

$$[L^{p_0}(\Omega), L^{p_1}(\Omega)]_\theta = L^p(\Omega), \text{ where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and their norms coincide. (In the case where $p_0 = \infty$ or $p_1 = \infty$ the statement is correct as long as we adopt the convention $\frac{1}{\infty} = 0$).

Conclusion

To conclude this project on the Riesz-Thorin theorem, let us sum up what we saw throughout our work.

We began with the proof of the theorem itself, thanks to an intermediate result known as the "Three lines theorem".

We then used this result to get various properties on convolutions, the Fourier transform, and also inequalities on the norm of functions of L^p spaces.

Finally, we extended the concept of interpolation to Banach spaces, and explained how one can proceed to do so.

The final example of this document showed how the interpolation space of L^{p_0} and L^{p_1}, where $1 \leq p_0, p_1 \leq +\infty$, is also an L^p space, where $\frac{1}{p}$ is a convex combination of $\frac{1}{p_0}$ and $\frac{1}{p_1}$.

This shows that the Riesz-Thorin theorem is none other than a specific case of space interpolation, where the spaces are L^p spaces !