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Local Convergence of Random Graphs

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I Introduction

I.1 Motivation and outline

In this document, we rigorously define the notion of a local limit for (possibly random) graphs. We explore several properties and criteria to practically determine the local limit of different models. The primary goal of this manuscript is to establish the theoretical foundations necessary for a more detailed study of certain random graphs. Specifically, we aim to connect the computation of balance indices with the local limit (see e.g., [Bie+24]).

The study of random graphs lies at the intersection of probability theory and graph theory. This branch of mathematics is increasingly used to address questions about the structure of complex networks modeling various real-world phenomena, from Internet networks to evolutionary biology.

The notion of local limit was introduced by Benjamini and Schramm (2001), and independently by Aldous and Steele (2004) in a different context. It formalizes the intuitive idea that a finite graph, viewed from the perspective of a randomly chosen vertex, resembles a limiting graph when observed at a local scale.

Over the years, local convergence has proven to be a powerful and elegant tool for understanding local behavior of large random structures. It provides a framework for comparing graphs at a local level, focusing on the structure of neighborhoods around individual vertices, rather than global properties such as connectivity. This approach has been crucial for understanding the asymptotic behavior of random graphs, particularly in contexts where local structure significantly influences the overall properties of the graph.

For example, local convergence to a limiting tree (a common occurrence in random graphs) is referred to as locally tree-like behavior. These trees are often described by branching processes (such as for the Erdős–Rényi random graph). Since trees are generally simpler structures than graphs, understanding the limiting tree often suffices to understand the behavior of the random graph. In summary, local convergence is a central technique in random graph theory, as many properties of random graphs are determined by their local limits.

This document builds upon the second volume of Random Graphs and Complex Networks by Remco van der Hofstad [Hof24]. We begin with some reminders on graphs and then introduce the notion of local convergence, first in the context of graphs, followed by a more general approach applicable to any metric space. Beyond the mathematical satisfaction that such a generalization provides, it also offers several practical advantages. For example, certain proofs become clearer once essential properties are abstracted. Additionally, the local convergence of marked graphs necessitates this generalization, as the marks take values in a metric space independent of the graph in question.

We proceed by providing general definitions of local convergence in various probabilistic senses, along with criteria (specifically, the equivalence to the appropriate convergence of subgraph counts) for determining local limits in specific cases. We conclude with several examples, both deterministic and random. Beyond their illustrative value, the second-moment method used could later be applied to study other properties of random graphs.

This work was conducted under the supervision of Dr. François Bienvenu and Dr. Jean-Jil Duchamps. I thank them for their time, for answering all my questions as I learned about these concepts, and their help in proofreading this manuscript.

I.2 Graphs: definitions and first properties

Definition I.2.1 ***Graph***

A graph $G = (V, E)$ consists of a collection $V = V(G)$ of vertices, called the vertex set, and a collection of edges $E = E(G)$, called the edge set. The vertices correspond to the objects that we model; the edges indicate some relation between pairs of these objects. We will assume that the vertex set is countable.

Remark:

- In this document (see definition I.2.1) graphs will usually be considered *undirected*. Thus, an edge is an unordered pair $\{u, v\} \in E$ indicating that u and v with $u, v \in V(G)$ are adjacent (or directly connected). When G is undirected, if u is directly connected to v then v is also directly connected to u. Therefore, an edge can be seen as a pair of vertices.
- We sometimes work with multi-graphs, which are graphs where self-loops or multiple edges between vertices are allowed.
- We often deal with finite graphs and then, for simplicity, we often take $V = [n] := \{1, ..., n\}$.

Definition I.2.2 ***Planar graph***

A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints (the vertices). In other words, it can be drawn in such a way that no edges cross each other.

Example I.2.1

For instance the following picture represents a graph $G = (V(G), E(G))$ where:
$V(G) = \{v_1; ...; v_5\}$ and $E(G) = \{\{v_1, v_2\}, \{v_4, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}\}$.

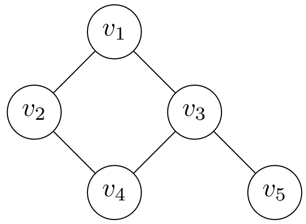

Figure 1: An example of a graph.

Definition I.2.3 ***Degree of a vertex***

The degree $d_u^{(G)}$ of a vertex $u \in V(G)$ in the graph G is equal to the number of edges containing u, i.e.

$$d_u^{(G)} = \#\{v \in V(G) : \{u, v\} \in E(G)\}.$$

Example I.2.2

For instance the degree of vertex v_3 in the previous example is 3.

Definition I.2.4 ***Degree distribution***

Let G be a finite graph. We let the degree distribution of G be $(p_k^{(G)})_{k\geqslant 0}$, where:

$$p_k^{(G)} = \frac{1}{|V(G)|} \sum_{v\in V(G)} \mathbf{1}_{\{d_v^{(G)}=k\}}.$$

Definition I.2.5 ***Walk, trail, path***

A walk of a graph $G = (V(G), E(G))$ is a finite sequence of vertices $\gamma = (v_1, ..., v_n)$ such that $\forall i \in [n-1],\ \{v_i, v_{i+1}\} \in E(G)$. A trail is a walk in which all edges are distinct. A path is a trail in which all vertices (and therefore also all edges) are distinct.

Example I.2.3

For instance, in the following graph we have a path between v_2 and v_5:

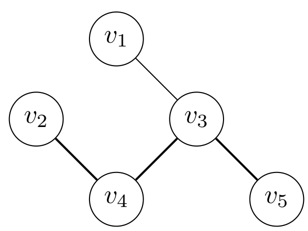

Figure 2: A path in a graph.

Definition I.2.6 ***Connected graph***

A graph is said to be connected if for all $u, v \in V(G)$ there exists a path from u to v.

Definition I.2.7 ***Circuit, cycle***

A circuit is a non-empty trail in which the first and last vertices are equal (closed trail). A cycle is a circuit in which only the first and last vertices are equal.

Definition I.2.8 ***Tree***

A tree is a connected graph that contains no cycle.

Definition I.2.9 ***Rooted Graph***

A rooted graph is a pair (G, o) where $G = (V(G), E(G))$ is a graph with vertex set $V(G)$, edge set $E(G)$ and a root vertex $o \in V(G)$. Moreover a graph is said to be locally finite when each of its vertices has finite degree (note that the degrees do not have to be uniformly bounded).

Example I.2.4

The first graph we gave (see Figure 1) was not a tree because it contained a cycle; but the second (see Figure 2) is. In the case of trees, it is common to treat the edges as pointing away from the root and to draw the tree with the root on top and the edges pointed downwards, as in Figure 3. The vertices on the bottom of an edge are called the *children* of the vertex on the top of the edge. For instance v_4 and v_5 are the children of v_3 in figure 3.

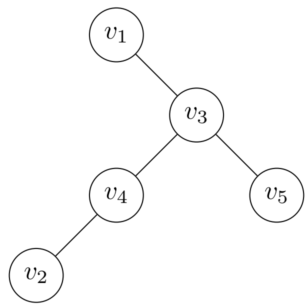

Figure 3: Conventional representation of a tree.

Definition I.2.10 ***Neveu-Ulam-Harris tree***

A plane (locally finite) rooted tree can be "embedded" in a canonical way in the Neveu-Ulam-Harris tree defined as follows:
Write $\mathcal{N}$ for the tree with vertex set $V(\mathcal{N}) = \bigcup_{n \geqslant 0} \mathbb{N}^n$ (with the convention $\mathbb{N}^0 = \{\varnothing\}$ where $\varnothing$ is the root of the tree and for $v \in V(\mathcal{N})$ a vertex, the elements $v0, v1, ...$ are the children of v ordered as follows:

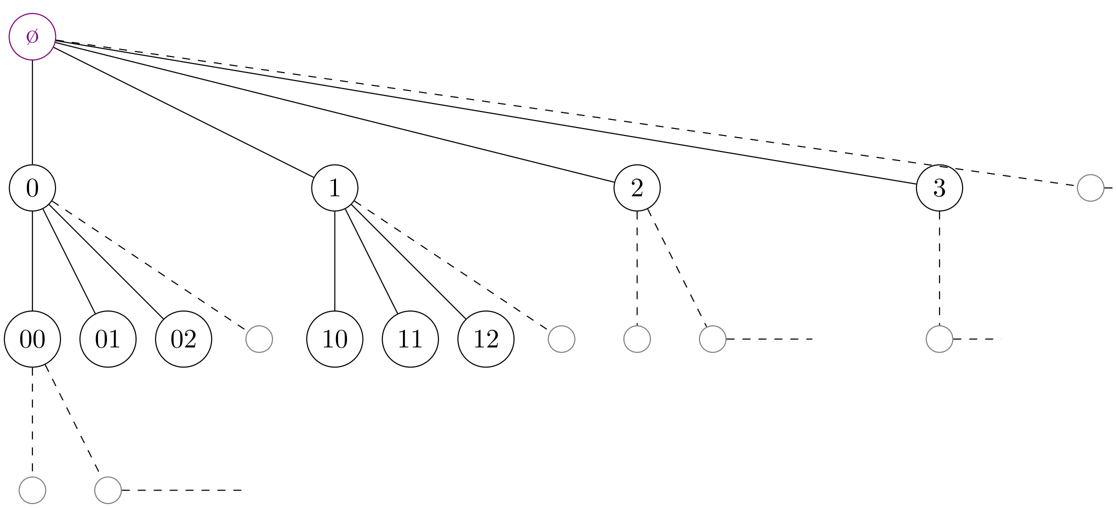

Figure 4: Representation of the Neveu-Ulam-Harris Tree.

The canonical embedding of a tree T is the injection $\Phi : T \to \mathcal{N}$ such that :

- $\Phi(\rho) = \varnothing$ (where ρ is the root of T).
- If $v \in V(T)$ and $(u_0, u_1, ..., u_k, ...)$ are its (ordered) children, then $\Phi(u_i) = \Phi(v) \cdot i$ for all

i where $\cdot$ stands for the concatenation operator (i.e. $(a,b,c)\cdot d=(a,b,c,d)$).

Example I.2.5

For instance the tree in figure 3 can be encoded with the Neveu-Ulam-Harris notation as follows:

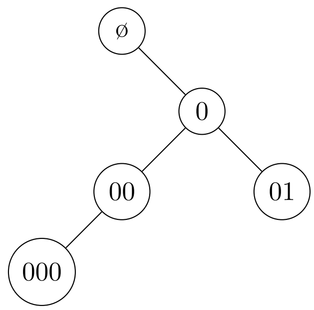

Figure 5: The Neveu-Ulam-Harris encoding.

Definition I.2.11 ***Graph distance***

For $(u,v)\in V(G)^2$, $\mathrm{dist}_G(u,v)$ denotes the length of one of the shortest path joining u and v.

Definition I.2.12 ***Subgraph***

For a rooted graph (G,o) we let $B_r^{(G)}(o)$ denote the (rooted) subgraph of (G,o) consisting of all vertices at graph distance at most $r\in\mathbb{N}$ away from o. Formally this means that:

$$B_r^{(G)}(o)=\left(V\left(B_r^{(G)}(o)\right),E\left(B_r^{(G)}(o)\right)\right)\text{, where:}$$
$$V\left(B_r^{(G)}(o)\right)=\{u:\ dist_G(o,u)\leqslant r\}\text{ and,}$$
$$E\left(B_r^{(G)}(o)\right)=\{\{u,v\}\in E:\ dist_G(o,u)\text{ and }dist_G(o,v)\leqslant r\}$$

Moreover, let $\partial B_r^{(G)}(o)$ denote the (unrooted) graph with:

$$\text{Vertex set: }V\left(\partial B_r^{(G)}(o)\right):=V\left(B_r^{(G)}(o)\right)\setminus V\left(B_{r-1}^{(G)}(o)\right)\text{, and}$$
$$\text{Edge set }E\left(\partial B_r^{(G)}(o)\right):=E\left(B_r^{(G)}(o)\right)\setminus E\left(B_{r-1}^{(G)}(o)\right)$$

Definition I.2.13 ***Connected component***

Let G be a graph and o be a vertex of G. We define the connected component of o in G as the maximal connected subgraph containing o.

Definition I.2.14 ***Graph Isomorphism***

- Two graphs are called isomorphic (denoted by $G_1\simeq G_2$) when there exists a bijection Φ mapping $V(G_1)$ to $V(G_2)$ such that $\{u,v\}\in E(G_1)$ precisely when $\{\Phi(u),\Phi(v)\}\in E(G_2)$.
- Similarly two rooted graphs $G_1(o_1)$ and $G_2(o_2)$ are isomorphic (denoted by $(G_1,o_1)\simeq$

(G_2, o_2)) when there exists a bijection Φ mapping $V(G_1)$ to $V(G_2)$ such that $\Phi(o_1) = o_2$ and $\{u, v\} \in E(G_1)$ precisely when $\{\Phi(u), \Phi(v)\} \in E(G_2)$.

Remark:

- The bijection implies that two isomorphic graphs G_1 and G_2 have the same number of vertices. Moreover, since $\{u, v\} \in E(G_1) \iff \{\Phi(u), \Phi(v)\} \in E(G_2)$, G_1 and G_2 also have the same number of edges.
- We let $\mathcal{G}_\star$ denote the space of rooted graphs and $[\mathcal{G}_\star]$ the space of rooted graph modulo isomorphism. When there is no possible confusion, we omit the equivalence classes and write $(G, o) \in \mathcal{G}_\star$ instead of $[(G, o)] \in [\mathcal{G}_\star]$.
- These definitions extend to multigraphs by requiring Φ to satisfy $x_{u,v} = x'_{\Phi(u),\Phi(v)}$ for every $\{u, v\} \in E(G)$ where $x_{u,v}$ denotes the number of edges (and $x_{u,u}$ the number of self-loops).

Proposition I.2.1 ***Degree distribution and isomorphism***

Let G_1 and G_2 be two finite graphs. If $G_1 \simeq G_2$, then G_1 and G_2 have the same degree distribution.

Proof:

Let $v \in V(G_1)$. Then $\Phi(v) \in V(G_2)$. Moreover, since for all $u \in E(G_1)$
$\{u, v\} \in E(G_1) \iff \{\Phi(u), \Phi(v)\} \in E(G_2)$, we thus have $d_v^{(G_1)} = d_{\Phi(v)}^{(G_2)}$. We conclude that for all $k \geqslant 0, p_k^{(G_1)} = p_k^{(G_2)}$ so G_1 and G_2 indeed have the same degree distribution.

$\blacksquare$

II The topology of local convergence

II.1 Local convergence of graphs

In this paragraph, we aim at defining a topology on the space of (rooted) graphs that underlines their local properties.

Definition II.1.1 ***Graph local distance***

Let (G_1, o_1) and (G_2, o_2) be two connected graphs and write $B_r^{G_i}(o_i)$ for the r-neighborhood of o_i in G_i, i.e. for the closed ball of radius r around o_i in G_i.
Let $R^\star := \sup\{r : \; B_r^{(G_1)}(o_1) \simeq B_r^{(G_2)}(o_2)\}$ and define the following:

$$d_{\mathcal{G}_\star}\left((G_1, o_1), (G_2, o_2)\right) := \frac{1}{R^\star + 1},$$

with the convention $\frac{1}{\infty} = 0$.

Remark:

- In this definition, the fact that $x \in \mathbb{R}_+ \mapsto \frac{1}{1+x}$ is decreasing is what matters. We could have choose any decreasing function (though this one will appear to be coherent with the generalization made in section II.2).
- Let (G, o) be a rooted graph and $r \geqslant 1$. Then $d_{\mathcal{G}_\star}(B_r^{(G)}(o), (G, o)) \leqslant \frac{1}{r+1}$. Indeed:

  $$\begin{aligned} d_{\mathcal{G}_\star}(B_r^{(G)}(o), (G, o)) \leqslant \frac{1}{r+1} &\iff \sup\{s : B_s^{(B_r^{(G)})}(o) \simeq B_r^{(G)}(o)\} \geqslant r \\ &\iff \forall s \leqslant r, \; B_s^{(B_r^{(G)})}(o) \simeq B_r^{(G)}(o) \end{aligned}$$

  and the last line comes from the fact that $B_s^{(B_r^{(G)})}(o) = B_r^{(G)}(o)$ for $s \leqslant r$.
  Moreover, the equality holds when $B_r^{(G)}(o) = B_{r+1}^{(B_r^{(G)})}(o) \not\simeq B_r^{(G)}(o)$, thus it holds when $B_r^{(G)}(o) \neq B_{r+1}^{(G)}(o)$ i.e. when it exists at least one vertex at distance $r+1$ of the root.
- In order to show that $d_{\mathcal{G}_\star}$ is indeed a metric on the space of rooted graph, we will need the following lemma.

Lemma II.1.1 ***Local neighborhoods determine the graph***

Let (G_1, o_1) and (G_2, o_2) be two connected locally finite rooted graphs such that for all $r \geqslant 0$, $B_r^{(G_1)} \simeq B_r^{(G_2)}$. Then $(G_1, o_1) \simeq (G_2, o_2)$.

Proof:

We aim at constructing ψ an isomorphism between (G_1, o_1) and (G_2, o_2) using the following :
Let $r \geqslant 0$, by assumption there exists $\phi_r : B_r^{(G_1)} \longrightarrow B_r^{(G_2)}$ an isomorphism and we can extend it to G_1 by defining:

$$\psi_r : v \in G_1 \longmapsto \begin{cases} \phi_r(v) & \text{if } v \in B_r^{(G_1)} \\ o_2 & \text{else.} \end{cases}$$

Write $V_r^{(G_1)}$ for $V\left(B_r^{(G_1)}\right)$. For every $r \geqslant 0$, $\phi_r|_{V_1^{(G_1)}}$ is an isomorphism between $B_1^{(G_1)}$ and

$B_2^{(G_2)}$. Since they are finite spaces, there are only finitely many such ismorphisms so one is common to an infinity of values of r. Let ϕ'_1 be such an isomorphism and let A_1 be the set of values of r for which $\phi_r|_{V_1^{(G_1)}} = \phi'_1$.

Now we extend this argument for $k = 2$ to the sequence $\left(\phi_r|_{V_2^{(G_1)}}\right)_{r\in A_1}$.

It exists ϕ'_2 and A_2 such that A_2 is infinite and $\phi_r|_{V_2^{(G_1)}} = \phi'_2$ for all $r \in A_2$. We then generalize this construction to $k \geqslant 2$ and the constructed sequence $(A_k)_{k\in\mathbb{N}}$ is a decreasing sequence of infinite sets. Denote $U_0 = \{o_1\}$ and $U_k = V_k^{(G_1)} \setminus V_{k-1}^{(G_1)}$. Thus $G_1 = \bigsqcup_{k\geqslant 0} U_k$ because G_1 is connected.

Let's then define $\psi : v \longmapsto \sum_{k\geqslant 1} \phi'_k(v)\mathbb{1}_{\{v\in U_k\}}$

Then ψ is an isomorphism between (G_1, o_1) and (G_2, o_2). Indeed ψ is bijective since $\phi'_k : U_k \longrightarrow \phi'_k(U_k)$ is bijective (because $\forall k \leqslant l,\ \phi'_k(v) = \phi'_l(v),\ \forall v \in V_k^{(G_1)}$).

Moreover, denote $k = \max\{dist_{G_1}(o_1, u), dist_{G_1}(o_1, v)\}$ for $(u, v) \in V(G_1)^2$ fixed. Then $u, v \in V_k^{(G_1)}$ and since ϕ'_k is an isomorphism between $B_k^{(G_1)}$ and $B_k^{(G_2)}$ we have that:

$(\psi(u) = \phi'_k(u), \psi(v) = \phi'_k(v)) \in V_k^{(G_2)} \times V_k^{(G_2)} \subset V^{(G_2)} \times V^{(G_2)}$ and $\{\psi(u), \psi(v)\} \in E_k^{(G_2)}$ if and only if $\{u, v\} \in E_k^{(G_1)}$ (where $E_k^{(G_i)} = E(B_k^{(G_i)})$). Thus,

$\{\psi(u), \psi(v)\} \in E^{(G_2)}$ if and only if $\{u, v\} \in E^{(G_1)}$. Finally $\phi(o_1) = o_2$ which completes the proof.

■

We thus have the following property:

Proposition II.1.1 $d_{\mathcal{G}_\star}$ ***is ultrametric***

The map $d_{\mathcal{G}_\star} : \mathcal{G}_\star \times \mathcal{G}_\star \longrightarrow [0; 1]$ is ultrametric, meaning that:

1. $d_{\mathcal{G}_\star}((G_1, o_1), (G_2, o_2)) = 0 \Longleftrightarrow (G_1, o_1) \simeq (G_2, o_2)$
2. $d_{\mathcal{G}_\star}((G_1, o_1), (G_2, o_2)) = d_{\mathcal{G}_\star}((G_2, o_2), (G_1, o_1))$
3. $d_{\mathcal{G}_\star}((G_1, o_1), (G_3, o_3)) \leqslant \max\{d_{\mathcal{G}_\star}((G_1, o_1), (G_2, o_2)), d_{\mathcal{G}_\star}((G_2, o_2), (G_3, o_3))\}$

Proof:

1. We have that,

$$\begin{aligned} d_{\mathcal{G}_\star}((G_1, o_1), (G_2, o_2)) = 0 &\Longrightarrow B_r^{(G_1)} \simeq B_r^{(G_2)},\ \forall r \geqslant 0 \\ &\Longrightarrow G_1 \simeq G_2 \qquad\qquad \text{(by Lemma II.1.1)} \end{aligned}$$

2. This follows from the definition.
3. If $B_r^{(G_1)} \simeq B_r^{(G_2)}$ and $B_r^{(G_2)} \simeq B_r^{(G_3)}$, then $B_r^{(G_1)} \simeq B_r^{(G_3)}$.
   Thus, $\sup\{r \geqslant 0, B_r^{(G_1)} \simeq B_r^{(G_3)}\} \geqslant \min\left(\sup\{r \geqslant 0, B_r^{(G_1)} \simeq B_r^{(G_2)}\}, \sup\{r \geqslant 0, B_r^{(G_2)} \simeq B_r^{(G_3)}\}\right)$
   Hence the result.

■

Remark:

- We now want to show that $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is a Polish space. Notice that if $\tilde{G}_1 \simeq G_1$ and $\tilde{G}_2 \simeq G_2$ then by transitivity, for all $r \geqslant 0$, $B_r^{(G_1)} \simeq B_r^{(G_2)}$ if and only if $B_r^{(\tilde{G}_1)} \simeq B_r^{(\tilde{G}_2)}$. Therefore $d_{\mathcal{G}_\star}$ is independant of the choice of the representative so $d_{\mathcal{G}_\star}$ is a well-defined metric on $[\mathcal{G}_\star]$.
- First, let us introduce a lemma critical to the proof of the completeness.

Definition II.1.2 ***Compatibility***

We say that a sequence of connected locally finite rooted graphs $((G_r, o_r))_{r \geqslant 0}$ is compatible when $B_r^{(G_s)}(o_s) \simeq (G_r, o_r)\ \forall r \leqslant s$.

Lemma II.1.2 ***Coherence property***

Let $((G_r, o_r))_{r \geqslant 0}$ be connected locally finite rooted graphs that are compatible. Then there exists a connected locally finite rooted graph (G, o) such that $(G_r, o_r) \simeq B_r^{(G)}(o)$. Moreover, (G, o) is unique up to isomorphism.

Proof:

First we need to construct a version $(G'_r, o'_r) \simeq (G_r, o_r)$ that has a common vertex set and then we will be able to define (G, o).
To construct it, we have to be careful that we do it in a compatible way. Write $V_r = [N_r]$ where $N_r = \#V(B_r^{(G_r)}(o_r))$. We then define $\phi_r : V(B_r^{(G_r)}(o_r)) \longrightarrow V_r$ recursively as follows : $\phi_0 : o_0 \longmapsto 1$.
Then let ψ_r be an isomorphism between (G_{r-1}, o_{r-1}) and $B_{r-1}^{(G_r)}(o_r)$ and η_r a bijection between $V(G_r) \setminus V(B_{r-1}^{(G_r)}(o_r))$ to $V_r \setminus V_{r-1}$ (which exists because of equality of cardinals). Then, we define:

$$\phi_r : v \longmapsto \begin{cases} \phi_{r-1}(\psi_r^{-1}(v)) & \text{if } v \in B_{r-1}^{(G_r)}(o_r) \\ \eta_r(v) & \text{otherwise.} \end{cases}$$

Thus, ϕ_r is a bijection between $V(G_r)$ and V_r and we can define $(G'_r, o'_r) = (\phi_r(G_r), \phi_r(o_r))$.
Then $\forall r \geqslant 0$ $o'_r = 1$ and ϕ_r is a bijection so $B_r^{(G'_r)}(o_r) \simeq B_r^{(G_r)}(o_r) \simeq (G_r, o_r)$ (by compatibility).
We can thus define (G, o) such as:

$$o = 1, \quad V(G) = \bigcup_{r \geqslant 0} V(G'_r) \text{ and } E(G) = \bigcup_{r \geqslant 0} E(G'_r).$$

Thus, $B_r^{(G)}(o) \simeq B_r^{(G'_r)}(o'_r) \simeq B_r^{(G_r)}(o_r) \simeq (G_r, o_r)$ as required. Further, (G, o) is locally finite since (G_r, o_r) is for every $r \geqslant 0$.
Finally if (G', o') also satisfies $B_r^{(G')}(o') \simeq (G_r, o_r)$, $\forall r \geqslant 0$, then $B_r^{(G')}(o') \simeq B_r^{(G)}(o)$, $\forall r \geqslant 0$ thus $(G', o') \simeq (G, o)$ by Lemma II.1.1, hence the uniqueness up to isomorphism.

■

We are now ready to prove that $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is Polish.

Proposition II.1.2 ***Completeness***

The metric space $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is complete.

Proof:

Let $([G_n, o_n])_{n\in\mathbb{N}}$ be a Cauchy sequence. Since $d_{\mathcal{G}_\star}$ is independent of the choice of the representative, we know that we can work with a representative sequence $((G_n, o_n))_{n\in\mathbb{N}}$ instead. Thus:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n, m \geqslant N,\ d_{\mathcal{G}_\star}(G_n, G_m) \leqslant \epsilon$$

Let $R^\star_{n,m}$ be the integer such that $d_{\mathcal{G}_\star}(G_n, G_m) = \frac{1}{R^\star_{n,m}+1}$. Then $\forall r \leqslant R^\star_{n,m},\ B_r^{(G_n)}(o_n) \simeq B_r^{(G_m)}(o_m)$. Thus we have equivalently that:

$$\forall r \geqslant 0,\ \exists n_r \in \mathbb{N},\ \text{such that } \forall n \geqslant n_r,\ B_r^{(G_n)}(o_n) \simeq B_r^{(G_{n_r})}(o_{n_r})$$

By defining recursively $n'_1 = n_1$ and $n'_k = \max\{n_1, ..., n_k\} + k$ we can assume that (n_r) is an increasing sequence. We can then define $(G'_r, o'_r) = B_r^{(G_{n_r})}(o_{n_r})$.

(G'_r, o'_r) forms a compatible sequence because for all $r \geqslant s,\ B_r^{(G'_s)} = B_r^{B_r^{(G_{n_s})}} = B_r^{(G_{n_s})} \simeq B_r^{(G_{n_r})} = G'_r$, and so by Lemma II.1.2 there exists a locally finite rooted graph (G, o) such that: $\forall n \geqslant n_r,\ B_r^{(G)}(o) \simeq (G'_r, o'_r) \simeq B_r^{(G_{n_r})}(o_{n_r}) \simeq B_r^{(G_n)}(o_n)$. Thus,
$d_{\mathcal{G}_\star}([G, o], [G_n, o_n]) = d_{\mathcal{G}_\star}((G, o), (G_n, o_n)) \leqslant \frac{1}{r+1} \underset{r\to\infty}{\longrightarrow} 0$.

So $[G_n, o_n] \xrightarrow{d_{\mathcal{G}_\star}} [G, o]$ and $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is complete.

■

Remark:

Before proving the separability, note that there is a countable number of isomorphism classes of rooted graphs (G, o) with radius at most r (fixed). Indeed, a locally finite (rooted) graph of radius at most r is a finite graph. Moreover, the number of isomophism classes of graphs having a finite vertex set is finite. Thus, the number of isomophism classes of rooted graphs with radius at most r is countable. This implies that the number of graphs (modulo isomorphism) that can occur as r-neighborhood of a random rooted locally finite graph is countable.

Proposition II.1.3 ***Separability***

The metric space $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is separable.

Proof:

Consider the set of all finite rooted graphs. Fix $[G, o]$ with representative (G, o). Then $B_r^{(G)}(o)$ is finite for all $r \geqslant 0$ and $d_{\mathcal{G}_\star}\left((G, o), B_r^{(G)}(o)\right) \leqslant \frac{1}{r+1}$ so $B_r^{(G)}(o) \xrightarrow{d_{\mathcal{G}_\star}} (G, o)$.
Thus the space of isomorphism classes of finite rooted graphs is dense and countable thus $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is separable.

■

We have thus shown that $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is a Polish space.

II.2 Local convergence of general metric spaces

In this section, we aim at generalizing the results of Section II.1. Beyond the fact that more general results are always mathematically satisfying it also has practical applications. For instance Section III.3 on marked rooted graphs rests on this generalization. Moreover, the critical results of Section III (for instance the criteria for local convergence) also rest on theorems proven in this section.
Firstly, we recall some definitions.

Definition II.2.1 ***Metric and metric space***

Let χ be a set. A *metric* on χ is a function $d_\chi \mapsto [0, \infty[$ such that:

- $d_\chi(x, y) = 0 \iff x = y$ for all $x, y \in \chi$;
- $d_\chi(x, y) = d_\chi(y, x)$ for all $x, y \in \chi$;
- $d_\chi(x, z) \leqslant d_\chi(x, y) + d_\chi(y, z)$ for all $x, y, z \in \chi$.

Let χ be a space and d_χ a metric on it. Then (χ, d_χ) is called a *metric space*.

Example II.2.1

- Let G be a connected graph. Then $(G, dist_G)$ is a metric space.
- $d_{\mathcal{G}_\star}$ is a metric on the the space $\mathcal{G}_\star$ (this is proven in section II.1).

Definition II.2.2 ***Polish space***

Let (χ, d_χ) be a metric space.

1. We say that (χ, d_χ) is *complete* when every Cauchy sequence has a limit. Here, a Cauchy sequence is a sequence $(x_n)_{n\in\mathbb{N}^\star}$ with $x_n \in \chi$ such that, for every $\epsilon > 0$ there exists an $N \in \mathbb{N}^\star$ such that for all $n, m \geqslant N,\ d_\chi(x_n, x_m) \leqslant \epsilon$.
2. We say that (χ, d_χ) is *separable* if it contains a countable subset that is dense in (χ, d_χ). This means that there exists a countable set $\mathcal{A} \subseteq \chi$ such that, for every $x \in \chi$, there exists a squence $(a_n)_{n\in\mathbb{N}^\star}$ with $a_n \in \mathcal{A}$ such that $d_\chi(a_n, x) \to 0$.
3. A topological space χ is called *Polish* when there exists a metric d_χ that generates its topology such that the metric space (χ, d_χ) is separable and complete.

Definition II.2.3 ***Restriction***

Let (χ, d_χ) be a metric space. For every $r \geqslant 0$, a restriction to radius r is a continuous function $[.]_r : \chi \to \chi$ that satisfies the following properties:

- (**Closure**) For all $x \in \chi, [x]_r \in \chi$
- (**Compatibility**) $[[x]_r]_s = [x]_s \qquad \forall x \in \chi$ and $r \geqslant s$
- (**Coherence**) For each sequence $(x_n)_{n\in\mathbb{N}} \in \chi^{\mathbb{N}}$ satisfying $[x_i]_j = x_j$ for all $j \in [i]$, there exists a unique element $x \in \chi$ such that $[x]_r = x_r$ for all $r \geqslant 0$.

Example II.2.2

- For $x = (x_n)_{n\in\mathbb{N}}$ element of $\chi = \mathbb{R}^{\mathbb{N}}$ and $r \geqslant 0$, $[x]_r := (x_1, ..., x_r, 0, 0, ...)$ is a restriction.
- For a graph (G, o) element of $\chi = \mathcal{G}_\star$, and $r \geqslant 0$, $[(G, o)]_r := B_r^{(G)}(o)$ is a restriction.

Definition II.2.4 ***Local topology***

Let $[.]_r : \chi \mapsto \chi$ be a restriction. Define for $x, y \in \chi$,

$$d_{loc}(x, y) = \sum_{r\geqslant 1} \frac{d_\chi([x]_r, [y]_r) \wedge 1}{r(r+1)}$$

where $a \wedge b = \min\{a, b\}$.

Remark:

Once again, the fact that the series converges is what matters. One could have chosen to replace $r(r+1)$ by 2^r for instance and the induced topology would have been the same.

Proposition II.2.1 ***The local distance is a metric***

Let (χ, d_χ) be a metric space, then (χ, d_{loc}) is also a metric space.

Proof:

Let us show that d_{loc} is a metric: for all $x, y, z \in \chi$,

- $\forall r \geqslant 1, d_\chi([x]_r, [y]_r) \geqslant 0$ because d_χ is a metric so $d_{loc} \geqslant 0$. Moreover, $d_{loc}(x, y) \leqslant \sum_{r\geqslant 1} \frac{1}{r(r+1)} = 1$ so $d_{loc}(x, y) < \infty$
- $d_{loc}(x, y) = d_{loc}(y, x)$ because for all $r \geqslant 1$, $d_\chi([x]_r, [y]_r) = d_\chi([y]_r, [x]_r)$
- Moreover,

$$\begin{aligned} d_{loc}(x, y) = 0 &\iff \forall r \geqslant 1 \ d_\chi([x]_r, [y]_r) = 0 \\ &\iff \forall r \geqslant 1 \ [x]_r = [y]_r \\ &\iff x = y \text{ by uniqueness in the coherence property of a restriction.} \end{aligned}$$

- $\forall r \geqslant 1$, $d_\chi([x]_r, [y]_r) \leqslant d_\chi([x]_r, [z]_r) + d_\chi([z]_r, [y]_r)$ (because d_χ is a metric).
  Thus $d_{loc}(x, y) \leqslant d_{loc}(x, z) + d_{loc}(z, y)$
  Thus, d_{loc} is a metric on χ.

■

Remark:

- The metric d_χ may depend strongly on "far away" changes in the sense that $d_\chi(x, y)$ could be large even when $[x]_r = [y]_r$ for most small values of r. For instance, once could consider two bounded sequences of which the firsts N terms are equal but on the $(N+1)$th there is a large gap between the two, along with the metric $d(x, y) = \sup_{i \in \mathbb{N}} |x_i - y_i|$.

The metric d_{loc} does not suffer from such effects in that x and y are at distance at most $\frac{1}{R+1}$ when $[x]_r = [y]_r$ for all $r \leqslant R$.

- When we define $d_\chi([x]_r, [y]_r) = \mathbb{1}_{\{[x]_r \neq [y]_r\}}$, we obtain with $R^\star := \sup\{r : [x]_r = [y]_r\}$ that:

$$\begin{aligned}
d_{loc} &= \sum_{r \geqslant 1} \frac{d_\chi([x]_r, [y]_r) \wedge 1}{r(r+1)} \\
&= \sum_{r \geqslant R^\star + 1} \frac{1}{r(r+1)} \\
&= \sum_{r \geqslant R^\star + 1} \frac{1}{r} - \frac{1}{r+1} \\
&= \frac{1}{R^\star + 1}
\end{aligned}$$

Thus the topology we defined on rooted graphs can be seen as a special case of local topology, hence our interest for it.

Theorem II.2.1 ***Local topologies form a Polish space***

Assume that $(\{[x]_r : x \in \chi\}, d_\chi)$ is Polish for every $r \geqslant 1$. Then the space (χ, d_{loc}) is a Polish space. Moreover, a subset $\mathcal{A} \subseteq \chi$ is relatively compact (meaning that $\overline{\mathcal{A}}$ is compact) if and only if the sets $\{[x]_r : x \in \mathcal{A}\}$ are relatively compact.

Proof:

- We have already shown that (χ, d_{loc}) is a metric space.

- Let us show the separability of (χ, d_{loc}).
  Let $r \geqslant 1$, by hypothesis $(\{[x]_r, x \in \chi\}, d_\chi)$ is separable. Consider $\mathcal{A}_r$ a countable and dense set in $(\{[x]_r,\ x \in \chi\}, d_\chi)$. Let $\mathcal{A} := \bigcup_{r \geqslant 1} \mathcal{A}_r$. Then $\mathcal{A}$ is countable. We must now show that it is dense in (χ, d_{loc}):
  Let $x \in \chi,\ \epsilon > 0$ and pick $r \geqslant 1$ such that $\frac{1}{r+1} < \epsilon$.
  Moreover, $\forall \delta > 0,\ \exists y_\delta \in \mathcal{A}_r$ such that $d_\chi([x]_r, y_\delta) \leqslant \delta$.
  Thus by continuity of the restriction there exists $y \in \mathcal{A}$ such that for all $s \leqslant r,\ d_\chi([[x]_r]_s, [y]_s) \leqslant \epsilon$. Thus:

$$\begin{aligned}
d_{loc}(x, y) &\leqslant d_{loc}(x, [x]_r) + d_{loc}([x]_r, y) \\
&\leqslant \frac{1}{r+1} + \sum_{1 \leqslant s \leqslant r} \frac{d_\chi([[x]_r]_s, [y]_s)}{s(s+1)} + \sum_{s \geqslant r+1} \frac{1}{s(s+1)} \\
&\leqslant \epsilon + \epsilon \left(1 - \frac{1}{r+1}\right) + \epsilon \\
&\leqslant 3\epsilon
\end{aligned}$$

  Thus (χ, d_{loc}) is separable.

- Now, let us show the completeness of (χ, d_{loc}).

Let $(x_n)_{n\in\mathbb{N}} \subset (\chi, d_{loc})$ be a Cauchy sequence.
Then, for all $r \geqslant 1$, $([x_n]_r)_{n\in\mathbb{N}}$ is a Cauchy sequence for d_χ. Indeed, by contraposition:
If there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $p, q \geqslant N$ such that:

$$\begin{aligned} & 1 \geqslant d_\chi([x_p]_r, [x_q]_r) > \epsilon. \\ \text{Then,} \qquad & \frac{d_\chi([x_p]_r, [x_q]_r)}{r(r+1)} > \frac{\epsilon}{r(r+1)} \\ \text{So,} \qquad & \sum_{s\geqslant 1} \frac{d_\chi([x_p]_s, [x_q]_s)}{s(s+1)} > \frac{\epsilon}{r(r+1)} \end{aligned}$$

(and if $d_\chi([x_p]_r, [x_q]_r) > 1$ then $\displaystyle\sum_{s\geqslant 1} \frac{d_\chi([x_p]_s, [x_q]_s)}{s(s+1)} > \frac{1}{r(r+1)}$ so in both cases $(x_n)_{n\in\mathbb{N}}$ would not be a Cauchy sequence for d_{loc}).

Thus, by completeness, there exists y_r (our candidate to construct the limit) such that $([x_n]_r)_n$ converges to y_r (for d_χ). Moreover, by continuity of the restrictions:
$\forall s \leqslant r,\ [[x_n]_r]_s \longrightarrow [y_r]_s$ but by compatibility,
$[[x_n]_r]_s = [x_n]_s \longrightarrow y_s$, so by the uniqueness of the limit $[y_r]_s = y_s$ so by coherence of the restrictions, there exists $y \in \chi$ such that for all $r \geqslant 1,\ [y]_r = y_r$.
Now, let $\epsilon > 0$. Then,
$\exists r \geqslant 1$ such that $\frac{1}{r+1} < \epsilon$ and $N \in \mathbb{N}$ such that $\forall n \geqslant N,\ \forall s \leqslant r,\ d_\chi([x_n]_r, y_r) \leqslant \epsilon$
Thus,

$$\begin{aligned} d_{loc}(x_n, y) &= \sum_{1\leqslant s\leqslant r} \frac{d_\chi([x_n]_s, [y]_s) \wedge 1}{s(s+1)} + \sum_{s\geqslant r+1} \frac{d_\chi([x]_s, [y]_s) \wedge 1}{s(s+1)} \\ &\leqslant \epsilon + \epsilon \\ &\leqslant 2\epsilon \end{aligned}$$

So x_n converges toward y (for d_{loc}) thus (χ, d_{loc}) is complete.

- Since (χ, d_{loc}) is a complete, separable metric space, it is indeed a Polish space.

- We now show that: $\mathcal{A}$ is relatively compact implies that each $\{[x]_r : x \in \mathcal{A}\}$ is relatively compact. Let r be a positive integer. Since $[.]_r$ is a continuous function, $\{[x]_r : x \in \overline{\mathcal{A}}\}$ is a compact set. Thus $\{[x]_r : x \in \mathcal{A}\}$ is relatively compact (included in a compact).

- Finally we show that if for all $r \geqslant 1$, $\{[x]_r : x \in \mathcal{A}\}$ are relatively compact then $\mathcal{A}$ is relatively compact.
  Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{A}$. Then by hypothesis:
  $([x_n]_1)_n$ has a convergent subsequence: $([x_{\phi_1(n)}]_1)_n$.
  Similarly, $([x_{\phi_1(n)}]_2)_n$ has a convergent subsequence: $([x_{\phi_2\circ\phi_1(n)}]_2)_n$.
  By a diagonal argument, there exists ϕ (defined by $\phi(n) = \phi_n \circ ... \circ \phi_1(n)$) such that $(x_{\phi(n)})_n$ converges.
  Thus $\mathcal{A}$ is indeed relatively compact and this concludes the proof.

■

Definition II.2.5 ***Tightness of measures***

We recall that for a set M, a family $\mathcal{M} = (\mu_m)_{m\in M}$ of probability measures is said to be tight when:

$$\forall \epsilon > 0,\ \exists K_\epsilon \text{ a compact such that } \mu(K_\epsilon) \geqslant 1-\epsilon \text{ for all } \mu \in \mathcal{M}$$

Theorem II.2.2 ***Convergence of finite-dimensional distributions implies convergence in distribution***

Assume that $\{[x]_r :\ x \in \chi\}$ is Polish for every $r \geqslant 1$. The local topology satisfies the following properties:

1. A family $(X_i)_{i\in I}$ of random variables with values in (χ, d_{loc}) is tight if and only if the family $([X_i]_r)_{i\in I}$ is tight for every $r \geqslant 1$.
2. Let X_1 and X_2 be two random variables with values in (χ, d_{loc}) such that :
   $\mathbb{P}([X_1]_r \in \mathcal{A}) = \mathbb{P}([X_2]_r \in \mathcal{A})$ for every $\mathcal{A} \in \mathcal{B}_{loc}$ and any $r \geqslant 1$. Then $X_1 \overset{d}{=} X_2$.
   (where $\mathcal{B}_{loc}$ denotes the Borel σ-field on (χ, d_{loc}) i.e the σ-algebra generated by the open sets).
3. $(X_n)_{n\in\mathbb{N}}$ converges in distribution (in the local topology) towards a random variable X having law μ when for every $r \geqslant 1$ and Borel set $\mathcal{A} \in \mathcal{B}_{loc}$,
   $\mathbb{P}([X_n]_r \in \mathcal{A}) \longrightarrow \mathbb{P}([X]_r \in \mathcal{A})$.

Proof:

1. • Assume that $(X_i)_{i\in I}$ is tight. Then,
   for all $\epsilon > 0,\ \exists K_\epsilon$ compact of χ such that for all $i \in I : \mathbb{P}(X_i \in K_\epsilon) \geqslant 1-\epsilon$
   By Theorem II.2.1, $[K_\epsilon]_r$ is compact for all $r \geqslant 1$ and thus (since $\{X_i \in K_\epsilon\} \subset \{[X_i]_r \in [K_\epsilon]_r\}$):
   $\mathbb{P}([X_i]_r \in [K_\epsilon]_r) \geqslant 1-\epsilon$ for all $i \in I$ so $([X_i]_r)_{i\in I}$ is tight.
   • Conversely assume that for all $r \geqslant 1$, $([X_i]_r)_{i\in I}$ is tight. Then,
   $\forall \epsilon > 0, \exists K_{\epsilon,r}$ compact (of (χ, d_χ)) such that $\forall i \in I,\ \mathbb{P}([X_i]_r \notin K_{\epsilon,r}) \leqslant 2^{-r}\epsilon$.
   Write $A_\epsilon = \{x \in \chi,\ \forall r \geqslant 1\ [x]_r \in K_{\epsilon,r}\}$. Then, by Theorem II.2.1, A_ϵ is relatively compact. Write $K_\epsilon = \overline{A_\epsilon}$. Then K_ϵ is compact and since for all $i \in I$,
   $\{X_i \in A_\epsilon\} = \{\forall r \geqslant 1,\ [X_i]_r \in [K_\epsilon]_r\}$ we have that:
   $$\begin{aligned}\mathbb{P}(X_i \notin K_\epsilon) &\leqslant \mathbb{P}(X_i \notin A_\epsilon)\\ &= \mathbb{P}(\exists r \geqslant 1 \text{ such that, } [X_i]_r \notin K_{\epsilon,r})\\ &\leqslant \sum_{r=1}^{\infty} \mathbb{P}([X_i]_r \notin K_{\epsilon,r})\\ &= \sum_{r=1}^{\infty} \epsilon 2^{-r}\\ &= \epsilon\end{aligned}$$
   Thus $(X_i)_{i\in I}$ is tight.
2. Assume that $\mathbb{P}([X_1]_r \in A) = \mathbb{P}([X_2]_r \in A)$ for any $A \in \mathcal{B}_{loc}$ and any $r \geqslant 1$. Write :
   $\mathcal{M} = \{\{x \in \chi :\ [x]_r \in A\},\ A \in \mathcal{B}_{loc},\ r \geqslant 1\}$.
   Let us show that $\mathcal{M}$ generates the Borel field, i.e. $\sigma(\mathcal{M}) = \mathcal{B}_{loc}$.

- First, we need to show that $\sigma(\mathcal{M}) \subseteq \mathcal{B}_{loc}$.
  Write
  $$f_r : \left| \begin{array}{ccc} \chi & \longrightarrow & \chi \\ x & \longmapsto & [x]_r \end{array} \right.$$
  Then $\mathcal{M} = \bigcup_{r \geqslant 0} \{f_r^{-1}(A),\ A \in \mathcal{B}_{loc}\}$ so we need to show that f_r is $(\mathcal{B}_{loc},\ \mathcal{B}_{loc})$ measurable for all $r \geqslant 0$. It is thus sufficient to show that f_r is d_{loc}-continuous.
  Let $r \geqslant 0$. Let $x \in \chi$ and $\epsilon > 0$. We must find a $\delta > 0$ such that:
  $$\forall y \in \chi, \quad d_{loc}(x, y) \leqslant \delta \Rightarrow d_{loc}([x]_r, [y]_r) \leqslant 2\epsilon$$
  If $d_{loc}(x, y) \leqslant \delta$ then in particular $\frac{d_\chi([x]_r,[y]_r) \wedge 1}{r(r+1)} \leqslant \delta$ so for $\delta < \frac{1}{r(r+1)}$ we must have:
  $$d_\chi([x]_r, [y]_r) < \delta r(r+1)$$
  Moreover there exists $S > 0$ such that $\frac{1}{S+1} = \sum_{s \geqslant S+1} \dfrac{1}{s(s+1)} \leqslant \epsilon$
  Since f_s is d_χ-continuous (for all s) we have that there exists a $\delta > 0$ such that:
  $$d_{loc}(x, y) < \delta \Rightarrow d_\chi\left([[x]_r]_s, [[y]_r]_s\right) \leqslant \epsilon \quad \forall s \leqslant S$$
  Hence, $d_{loc}([x]_r, [y]_r) \leqslant \epsilon \left(1 + \sum_{s=1}^{S} \dfrac{1}{s(s+1)}\right) \leqslant 2\epsilon$
  Thus, f_r is d_{loc}-continuous so $\sigma(\mathcal{M}) \subseteq \mathcal{B}_{loc}$
- Now we must show that $\mathcal{B}_{loc} \subseteq \sigma(\mathcal{M})$.
  Write $\mathcal{F}_r := \{f_r^{-1}(A),\ A \in \mathcal{B}_{loc}\}$ and $\mathcal{F} = \sigma\left(\bigcup_{r \geqslant 0} \mathcal{F}_r\right) = \sigma(\mathcal{M})$.
  We need to show that open sets for d_{loc} are in $\mathcal{F}$. Since our space is separable (see theorem II.2.1) we only need to show that for all $x \in \chi$ and $\rho > 0$:
  $B_{loc}(x, \rho) := \{y :\ d_{loc}(x, y) < \rho\} \in \mathcal{F}$.
  For this, we need to show that $d_{loc}(x, .)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. In particular, it is sufficient to show that d_{loc} is $(\mathcal{F}^{\otimes 2}, \mathcal{B}(\mathbb{R}))$-measurable.
  Denote $g_r : (x, y) \mapsto (f_r(x), f_r(y))$. By limit and sum we only need to prove that for all $r \geqslant 0$, $d_\chi \circ g_r$ is $(\mathcal{F}^{\otimes 2}, \mathcal{B}(\mathbb{R}))$-measurable.
  Write $\mathcal{T}_r := \{f_r^{-1}(A),\ A \in \mathcal{B}_\chi\}$ and $\mathcal{T} = \sigma\left(\bigcup_{r \geqslant 0} \mathcal{T}_r\right)$.
  We are now going to show that $\mathcal{F} = \mathcal{T}$ by showing that for all $r \geqslant 0$, $\mathcal{F}_r = \mathcal{T}_r$. But by definition, $f_r : (\chi, d_{loc}) \to (\chi, d_\chi)$ is continuous so $f_r^{-1}(\mathcal{B}_\chi) \subset \mathcal{B}_{loc}$.
  But $f_r \circ f_r = f_r$ so since $\mathcal{B}_{loc} \subset \mathcal{B}_\chi$ we have that $f_r^{-1}(\mathcal{B}_\chi) \subset f_r^{-1}(\mathcal{B}_{loc}) \subset f_r^{-1}(\mathcal{B}_\chi)$.
  Thus $\mathcal{F} = \mathcal{T}$, so we only need to show that $d_\chi \circ g_r$ is $(\mathcal{T}^{\otimes 2}, \mathcal{B}(\mathbb{R}))$-measurable but this is obvious since by definition and continuity g_r is $(\mathcal{T}^{\otimes 2}, \mathcal{B}_\chi^{\otimes 2})$-measurable and by continuity (given by the reverse triangle inequality) d_χ is $(\mathcal{B}_\chi^{\otimes 2}, \mathcal{B}(\mathbb{R}))$-measurable.

Moreover $\mathcal{M}$ is clearly stable under finite intersections thus $X_1 \overset{d}{=} X_2$ by the $\lambda\!-\!\pi$ theorem.

3. Assume that for all $r \geqslant 1$ and $A \in \mathcal{B}_{loc}$, $\quad \mathbb{P}([X_n]_r \in A) \longrightarrow \mathbb{P}([X]_r \in A)$.
   In the previous point, we showed that $\mathcal{M}$ is stable under finite intersections and every open set of the local topology can be written as a countable union of sets in $\mathcal{M}$.
   Thus, $X_n \xrightarrow{d} X$.

■

Remark:

For $\chi = \mathcal{G}_\star$ and $d_\chi(G_1, G_2) = \mathbb{1}_{\{G_1 \simeq G_2\}}$ we have that (implicitly we use that B_r is a restriction) $d_{loc} = d_{\mathcal{G}_\star}$ so $d_{\mathcal{G}_\star}$ is well-defined on $\mathcal{G}_\star$.
Moreover this shows that theorem II.2.1 generalize our main result of section II.1, namely the fact that $([\mathcal{G}_\star], d_{\mathcal{G}_\star})$ is a Polish space.

III Local weak convergence of deterministic graphs

III.1 Definition and criterion

Definition III.1.1 ***Local weak convergence of deterministic graphs***

Let $G_n = (V(G_n), E(G_n))$ denote a finite connected graph. Let (G_n, o_n) be the rooted graph obtained by letting $o_n \in V(G_n)$ be chosen uniformly at random (u.a.r). We say that (G_n, o_n) converges locally weakly to the connected graph (G, o) which is a (possibly random) element of $\mathcal{G}_\star$ having law μ when, for every bounded and continuous function $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$,

$$\mathbb{E}[h(G_n, o_n)] \longrightarrow \mathbb{E}_\mu[h(G, o)]$$

(where the first expectation is with respect to the random vertex).
We write $(G_n, o_n) \xrightarrow{d} (G, o)$.

Remark:

To define continuity, one needs a topology. The one we take here is the local topology defined in section II.2 as one could guess from the name given to definition III.1.1.

Definition III.1.2 ***Extension to disconnected graphs***

For $v \in V(G_n)$, we let $\mathcal{C}(v)$ denote its connected component meaning that for o_n the root of G_n, we write $\mathcal{C}(o_n)$ for the rooted graph defined as:

$$V(\mathcal{C}(o_n)) = \{u : \ dist_{G_n}(o_n, u) < \infty\} \text{ and}$$
$$E(\mathcal{C}(o_n)) = \{\{u, v\} \in E(G_n) : \ dist_{G_n}(o_n, u), \ dist_{G_n}(o_n, v) < \infty\}$$

For $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$, by convention we extend the definiton above to all rooted locally finite graphs by letting: $h(G_n, o_n) \overset{def}{=} h(\mathcal{C}(o_n))$.

Before giving some criteria, we will need the following results:

Proposition III.1.1 ***Criterion for weak convergence***

A necessary and sufficient condition for $(\mathbb{P}_n)_{n\in\mathbb{N}}$ a sequence of probability measure to weakly converge to a certain $\mathbb{P}$ is that each subsequence contains a further subsequence converging weakly to $\mathbb{P}$.

Proof:

- The necessity part is clear.
- For sufficiency, if $\mathbb{P}_n$ does not weakly converges to $\mathbb{P}$ then there exists a continuous and bounded function f such that $\mathbb{E}_{\mathbb{P}_n}[f] \nrightarrow \mathbb{E}_{\mathbb{P}}[f]$ but then for some positive ϵ and some subsequence $\mathbb{P}_{n_i}$, $|\mathbb{E}_{\mathbb{P}_{n_i}}[f] - \mathbb{E}_{\mathbb{P}}[f]| > \epsilon$ for all i and thus no further subsequence can weakly converge to $\mathbb{P}$.

■

Definition III.1.3 ***Relative compactness***

A family $\mathcal{M} = (\mu_m)_{m\in M}$ of probability measures on a fixed metric space is said to be relatively compact if its closure under the Prokhorov metric (see e.g [Bil99]) is compact.

Remark:

Since the space of probability measures on a metric space is metrizable (by the Prokhorov metric, see e.g [Bil99]), a family $\mathcal{M}$ is relatively compact if and only if for every sequence of elements of $\mathcal{M}$ there exists a weakly convergent subsequence.

We recall the following standard theorem. See e.g. [Bil99] for a proof.

Theorem III.1.1 ***Prokhorov***

If $\mathcal{M}$ is tight, then it is relatively compact.

Proposition III.1.2 ***Laws of neighborhoods determine distribution***

Let μ and μ' be two distributions on $\mathcal{G}_\star$ such that $\mu(B_r^{(G)}(o) \simeq H^\star) = \mu'(B_r^{(G)}(o) \simeq H^\star)$ for all $r \geqslant 0$ and $H^\star \in \mathcal{G}_\star$. Then $\mu = \mu'$.

Proof:

We can rephrase the proposition as follows:
Let X_1 and X_2 be two random variables on $\mathcal{G}_\star$ such that $\mathbb{P}([X_1]_r = H^\star) = \mathbb{P}([X_2]_r = H^\star)$ then it is exactly theorem II.2.2.
In this case, we also give a more direct proof:
By definition $X_1 \overset{d}{=} X_2$ when $\mathbb{P}(X_1 \in \mathcal{H}_\star) = \mathbb{P}(X_2 \in \mathcal{H}_\star)$ for every measurable $\mathcal{H}_\star \subset \mathcal{G}_\star$. Fix $\mathcal{H}_\star \subset \mathcal{G}_\star$ and for $r \geqslant 0$, denote $\mathcal{H}_\star(r)$ the set containing a representative of each class of isomorphism of graphs that can occur as r-neighborhoods (i.e. all locally finite rooted graphs of radius at most r modulo isomorphism). By remark II.1 this set is countable.
Moreover by monotone continuity,

$$\mathbb{P}(X_1 \in \mathcal{H}_\star) = \mathbb{P}(\forall r \geqslant 1,\ [X_1]_r \in \mathcal{H}_\star(r)) = \lim_{r\geqslant 1} \searrow \mathcal{P}([X_1]_r \in \mathcal{H}_\star(r))$$

(and the same goes for X_2). Since,

$$\mathbb{P}([X_1]_r \in \mathcal{H}_\star(r)) = \sum_{H^\star\in\mathcal{H}_\star(r)} \mathbb{P}([X_1]_r \simeq H^\star) = \sum_{H^\star\in\mathcal{H}_\star(r)} \mathbb{P}([X_2]_r \simeq H^\star) = \mathbb{P}([X_2]_r \in \mathcal{H}_\star(r)),$$

we have that $\mathbb{P}(X_1 \in \mathcal{H}_\star) = \mathbb{P}(X_2 \in \mathcal{H}_\star)$.

$\blacksquare$

Theorem III.1.2 ***Criterion for local weak convergence***

The sequence of finite rooted graphs $(G_n, o_n)_{n\in\mathbb{N}}$ converges locally weakly to $(G, o) \sim \mu$ precisely when, for every rooted graph $H^\star \in \mathcal{G}_\star$ and all integers $r \geqslant 0$,

$$p_r^{(G_n)}(H^\star) = \frac{1}{\#V(G_n)} \sum_{v\in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v)\simeq H^\star\}} \underset{n\to\infty}{\longrightarrow} \mu(B_r^{(G)}(o) \simeq H^\star) \qquad (1)$$

where $B_r^{(G_n)}(v)$ is the rooted r-neighborhood of $v \in V(G_n)$ and $B_r^{(G)}(o)$ is is the rooted r-neighborhood of o in the limiting graph (G, o).

Proof:

- $\Rightarrow$: We notice that $p_r^{(G_n)}(H^\star) = \mathbb{E}[h_{H^\star,r}(G_n, o_n)]$ for $h_{H^\star,r} : (G, o) \longmapsto \mathbb{1}_{\{B_r^{(G)}(o) \simeq H^\star\}}$ and we only have to show that $h_{H^\star,r}$ is bounded and continuous:
It is clearly bounded, now let us show that it is continuous (for $r \geqslant 0$ and $H^\star$ fixed).
Let $0 < \epsilon < 1$ (since there is nothing to prove for $\epsilon \geqslant 1$), there exists $\delta > 0$ such that $\delta < \frac{1}{r+1}$. So when $d_{\mathcal{G}_\star}(G_1, G_2) < \delta,\ B_r^{(G_1)} \simeq B_r^{(G_2)}$. Then $B_r^{(G_1)} \simeq H^\star \iff B_r^{(G_2)} \simeq H^\star$, thus $|h_{H^\star,r}(G_1) - h_{H^\star,r}(G_2)| = 0 < \epsilon$. So $h_{H^\star,r}$ is indeed continuous and we have the direct implication.

- $\Leftarrow$: This result from the third point of theorem II.2.2. However, we will still give a more direct proof:
μ is a probability measure so $[(G_n, o_n)]_r$ converges in distribution by hypothesis hence is tight for every $r \geqslant 0$ Thus by the theorem II.2.2 (the first point), $((G_n, o_n))_{n\in\mathbb{N}}$ is tight so by *Prokhorov's theorem* it is relatively compact. Thus, for every subsequence there exists a further subsequence that converges in distribution. Denote $(G', o') \sim \mu'$ the limit. By hypothesis $\mu(B_r^{(G)}(o) \simeq H^\star) = \mu'(B_r^{(G')}(o') \simeq H^\star)$ (by passing to the limit) thus by the proposition III.1.2 we have that $(G', o') \sim \mu$.
Since this is true for every subsequence the local limit is indeed (G, o) (see prop III.1.1).

■

Remark:

Since we showed the convergence of each restriction (i.e the finite-dimensional distributions) we directly have the convergence of distribution but we chose to finish the proof using more classic results on tightness.

Theorem III.1.3 ***Local weak convergence and subsets***

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of rooted graphs and $o_n \in V(G_n)$ be a vertex chosen u.a.r. Let (G, o) be a random variable on $\mathcal{G}_\star$ having law μ. Let $\mathcal{T}_\star \subset \mathcal{G}_\star$ be a measurable subset of the space of rooted graphs. Assume that $\mu((G, o) \in \mathcal{T}_\star) = 1$. Then, $(G_n, o_n) \xrightarrow{d} (G, o)$ when Eq.(1) holds for all $H^\star \in \mathcal{T}_\star$ and all $r \geqslant 0$.

Proof:

We saw that the set $\mathcal{T}_\star(r)$ is countable.
Morevover, modulo isomophism $\bigcap\limits_{r\geqslant 0}(\searrow)\mathcal{T}_\star(r) = \mathcal{T}_\star$ thus, since $\mu((G, o) \in \mathcal{T}_\star) = 1$ we have that:
$\forall\epsilon,\ \exists r, m,\ \mathcal{T}_\star(r, m) \subset \mathcal{T}_\star(r)$ a subset of size at most m such that,

$$\mu(B_r^{(G)}(o) \in \mathcal{T}_\star(r, m)) \geqslant 1 - \epsilon.$$

Let $\epsilon > 0$ and fix such a $\mathcal{T}_\star(r,m)$. We have that:

$$\begin{aligned}\mathbb{P}(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)) &= 1 - \mathbb{P}(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r)) \\ &\leqslant 1 - \mathbb{P}(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r,m)) \qquad (\mathcal{T}_\star(r,m) \subset \mathcal{T}_\star(r))\end{aligned}$$

Thus,

$$\begin{aligned}\left(0 \leqslant \liminf_{n\to\infty} \mathbb{P}(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r))\right) &\leqslant \limsup_{n\to\infty} \mathbb{P}(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)) \\ &\leqslant 1 - \liminf_{n\to\infty} \mathbb{P}(B_r^{(G_n)}(o_n) \in \mathcal{T}_\star(r,m)) \\ &= 1 - \mu(B_r^{(G)}(o) \in \mathcal{T}_\star(r,m)) \\ &\leqslant \epsilon\end{aligned}$$

Thus, $\mathbb{P}(B_r^{(G_n)}(o_n) \notin \mathcal{T}_\star(r)) \underset{n\to\infty}{\longrightarrow} 0$, in particular for every $H^\star \notin \mathcal{T}_\star(r)$, we have that:

$$\mathbb{P}(B_r^{(G_n)}(o_n) \simeq H^\star) = p_r^{(G_n)}(H^\star) \longrightarrow 0 = \mu(B_r^{(G)}(o) \simeq H^\star)$$

Thus, if Eq(1) holds for all $H^\star \in \mathcal{T}_\star(r)$ then it holds for all $H^\star \in \mathcal{G}_\star$ hence the result (by theorem III.1.2).

■

Before giving more complex examples, let us start with simple ones.

Example III.1.1

- First we can give a simple example where the local weak limit of a sequence of deterministic graph is random:
  Consider the following sequence:
  $\forall n \in \mathbb{N},\ G_n = (V(G_n), E(G_n))$ where $V(G_n) = \{1,2,3\}$ and $E(G_n) = \{\{1,2\}\}$ meaning that the representation of G_n in the plane is as follows.

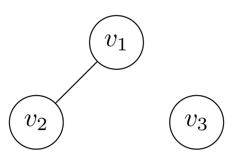

Figure 6: Representation of G_n.

  Denote H_1 the rooted graph with representation:

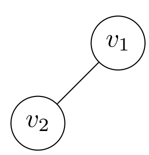

Figure 7: Representation of H_1.

  and H_2 the rooted graph with representation:

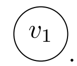

Figure 8: Representation of H_2.

Then $(G_n, o_n) \xrightarrow{d} (G, o) \sim \mu$ where $\mu(H_1) = \frac{2}{3}$ and $\mu(H_2) = \frac{1}{3}$
Indeed let X_n be a random variable such that $\mathbb{P}(X_n = (G_n, o_n)) = \frac{1}{3}$ $\forall o_n \in G_n$ and X a random variable such that $\mathbb{P}(X = H_1) = \frac{2}{3}$ and $\mathbb{P}(X = H_2) = \frac{1}{3}$.
For all $r \geqslant 1$, $p_r^{(G_n)}(H_1) = \frac{2}{3} \underset{n\to\infty}{\longrightarrow} \frac{2}{3} = \mathbb{P}(X = H_1)$ and $p_r^{(G_n)}(H_2) = \frac{1}{3} \underset{n\to\infty}{\longrightarrow} \frac{1}{3} = \mathbb{P}(X = H_2)$ (and it is obvious for $r = 0$) Thus $X_n \xrightarrow{d} X$ by theorem III.1.3.

- Before the general case, let us detail the example of two related graphs : a 'line' and a 'cycle'.
  Denote $(G, o) = (\mathbb{Z}, 0)$, $B_r^{(G)}(o) = (\llbracket -r; r \rrbracket,\ \{\{i, i+1\},\ i \in \llbracket -r; r-1 \rrbracket\})$.
  - <u>For the line:</u> Consider the graph G_n given by $V(G_n) = [n]$ and $E(G_n) = \{\{i, i+1\} : i \in [n-1]\}$. Let us show that (G_n, o_n) converges to (G, o) (where o_n is a vertex chosen u.a.r. in $V(G_n)$). By theorem III.1.3 it is sufficient to show that for all $r, s \geqslant 0$,
  $$p_r^{(G_n)}(H_s) = \frac{1}{\#V(G_n)} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H_s\}} \underset{n\to\infty}{\longrightarrow} \underbrace{\mathbb{P}(B_r^{(G)}(o) \simeq H_s)}_{= \begin{cases} 1 & \text{if } r = s \\ 0 & \text{else.} \end{cases}}$$
  where $H_s = B_s^{(G)}(o)$.
  Let $r, s \geqslant 0$.
    - If $r = s$:
      $\forall v \in [r, n-r]$, (with $n \geqslant 2r$), $B_r^{(G_n)}(v) \simeq H_s$. Thus,
      $p_r^{(G_n)}(H_s) = \frac{1}{n}(n - 2(r-1)) \underset{n\to\infty}{\longrightarrow} 1$
    - If $r \neq s$:
      Let $m = \max(r, s)$, $\forall v \in [m, n-m]$, (with $n \geqslant 2m$), $B_r^{(G_n)}(v) \not\simeq H_s$ (because the cardinality of the vertex sets differ). Thus $p_r^{(G_n)}(H_s) \underset{n\to\infty}{\longrightarrow} 0$, hence the result.
  - <u>For the cycle:</u> Consider the graph G_n given by $V(G_n) = [n]$ and $E(G_n) = \{\{i, i+1\} : i \in [n-1]\} \cup \{\{1, n\}\}$. We aim to show that the local limit of (G_n, o_n) also is (G, o). Similarly, we take interest in knowing when $B_r^{(G_n)}(v) \simeq H_s$ (for $r, s \geqslant 0$ and $v \in V(G_n)$).
  For $n \geqslant 2r + 1$ we see that for all $v \in V(G_n)$, $B_r^{(G_n)}(v) \simeq H_s$ if and only if $r = s$. Thus,
    - If $r = s$:
      For n large enough $p_r^{(G_n)}(H_s) = \frac{1}{n} \sum_{v \in [n]} 1 = 1 \underset{n\to\infty}{\longrightarrow} 1$
    - If $r \neq s$:
      For n large enough $p_r^{(G_n)}(H_s) = 0$ thus $p_r^{(G_n)}(H_s) \underset{n\to\infty}{\longrightarrow} 0$, Hence the result.

Even if globally, those two graphs are quite different, they "locally look alike" since from the perspective of a vertex, they have a similar structure. Their local limit being the same stresses this fact.

III.2 Examples of local weak convergence

III.2.1 Local weak convergence of Boxes in $\mathbb{Z}^d$

This is a generalization of example III.1.1 in d-dimensions. We let G_n be the graph such that $V(G_n) = \{(x_1, ..., x_d) \mid x_i \in [\![0; n]\!]\}$ and the edge $\{x, y\}$ is in $E(G_n)$ precisely when there is a unique $i \in [d]$ such that $|x_i - y_i| = 1$ and $\forall j \neq i,\ x_j = y_j$. Take a root u.a.r. and denote the resulting graph (G_n, o_n). Let o be a random vertex (fixed) of $\mathbb{Z}^d$. Then $(G_n, o_n) \xrightarrow{d} (\mathbb{Z}^d, o)$. Indeed by theorem III.1.3 we only need to show that $p_r^{(G_n)}(B_r^{(\mathbb{Z}^d)}(o)) \underset{n\to\infty}{\longrightarrow} 1$.

But $B_r^{(G_n)}(o_n) \simeq B_r^{(\mathbb{Z}^d)}(o)$ unless o_n is at distance less than r from one of the boundaries of $[n]^d$, meaning one of the coordinate is either in $[r-1]$ or in $[\![n-r+1; n]\!]$. But this event occurs with probability in $o(\frac{1}{n})$ hence the result (we used the same argument in the previous example).

III.2.2 Local weak convergence of truncated trees

Fix a degree d and a height n (that will later go to ∞). We define $\mathbb{T}_{d,n}$ the d-regular tree truncated at height n meaning that :

$$V(\mathbb{T}_{d,n}) = \{\varnothing\} \cup \bigcup_{k=1}^{n} [d] \times [d-1]^{k-1}$$

(with the Neveu-Ulam-Harris encoding). The edge set is as follows:
Let $v = v_1...v_k$ and $u = u_1...u_l$ two vertices. We say that u is the parent of v when $l = k-1$ and $u_i = v_i\ \forall i \in [k-1]$. We then define u and v as neighbors when u is the parent of v or vice-versa (see figure 9).
Now we choose o_n u.a.r. from $V(\mathbb{T}_{d,n})$ and we want to show that the local weak limit has the structure of the *canopy tree* $\mathbb{T}_d^c$ which is defined as follows:
Take $\mathbb{T}_{d,n}$ and choose a leaf at random. Make it the root (we call it the root-leaf) and then take n to ∞. The constructed graph is the so-called canopy tree.
From this root leaf there exists a unique infinite path. Let o_ℓ be the ℓ-th vertex on this infinite path. We can now consider $(\mathbb{T}_d^c, o_\ell)$ and we define the limiting measure μ by :

$$\mu((\mathbb{T}_d^c, o_\ell)) \overset{\text{def}}{=} \mu_\ell = (d-2)(d-1)^{-(\ell+1)}$$

meaning that for all $r \geqslant 0$, $\mu(B_r^{(G)}(o) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell)) = \mu_\ell$.

We can represent the situation with the following drawing (where o_2 represents a vertex chosen u.a.r at distance 2 from the closest leaf (here the rootleaf)):

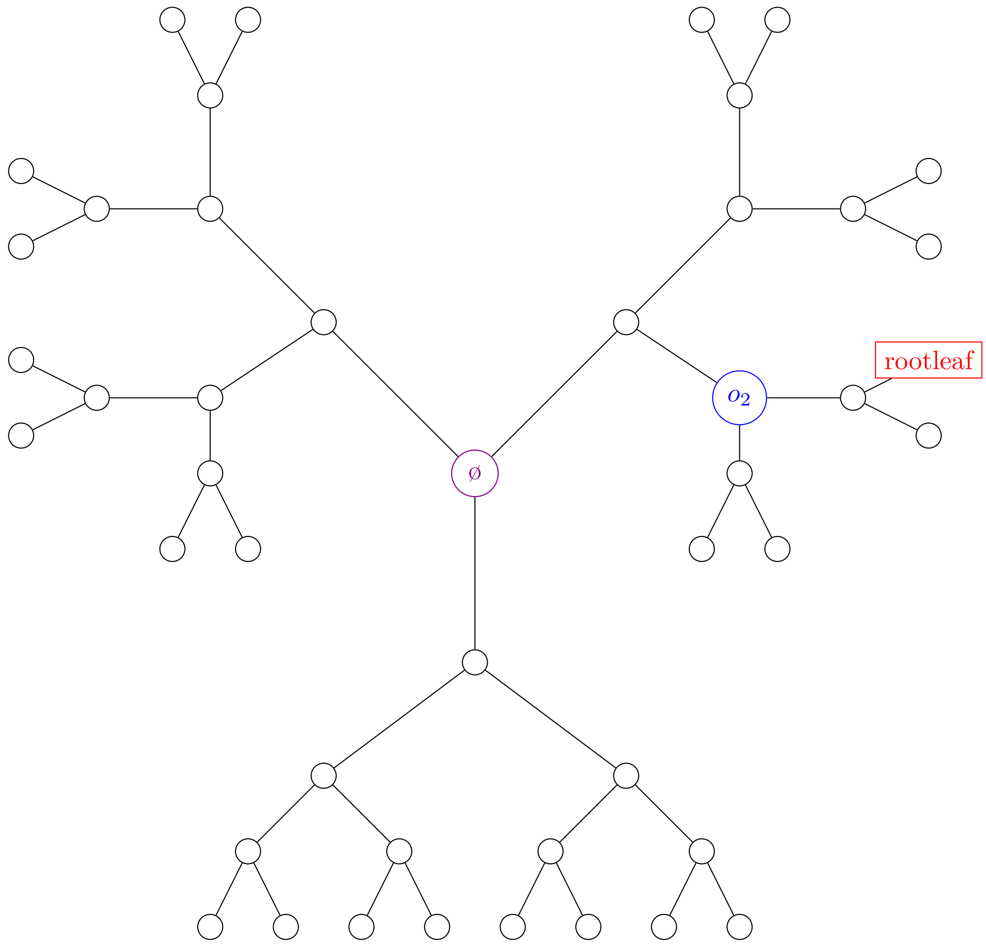

Figure 9: Representation of a truncated 3-regular tree of size 4.

Seen from the rootleaf perspective we have the following drawing:

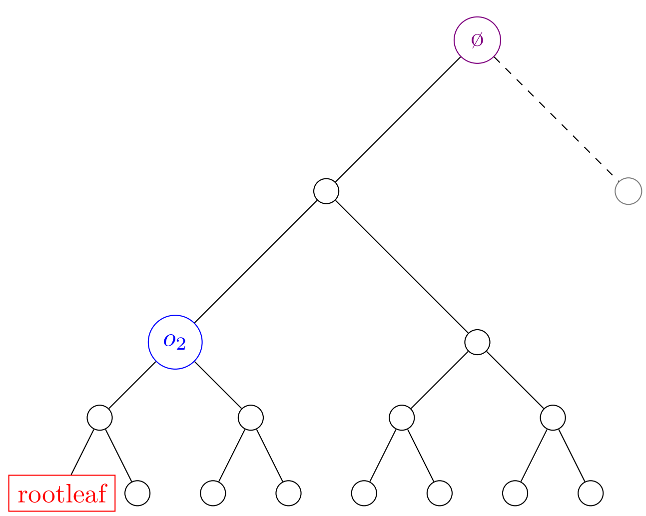

Figure 10: Representation from the root-leaf perspective.

As n goes to infinity we thus have the following:

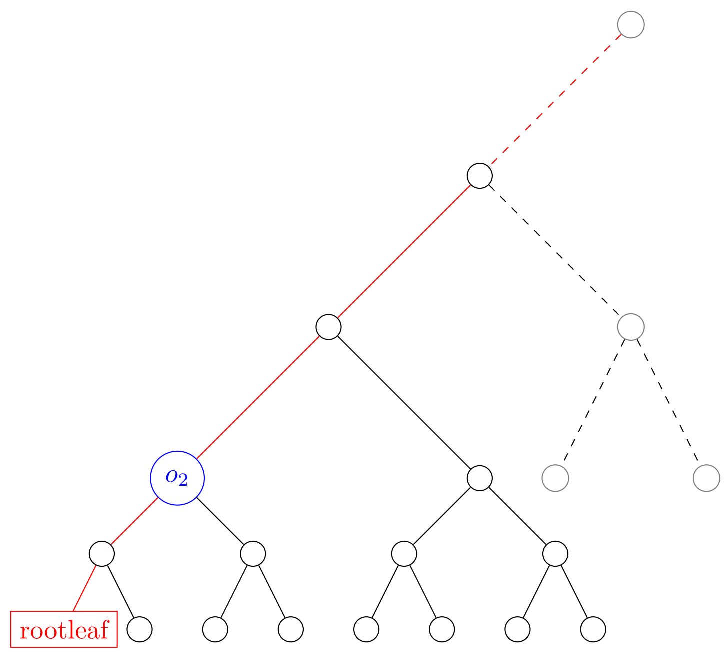

Figure 11: Representation of the canopy-tree.

where the unique infinite path from the rootleaf is highlighted in red.

We want to show that $(G_n, o_n) = (\mathbb{T}_{d,n}, o_n) \xrightarrow{d} (G, o) \sim \mu$. By theorem III.1.3 we only need to show that:

$$p_r^{(G_n)}(B_r^{(\mathbb{T}_d^c)}(o_\ell)) \underset{n\to\infty}{\longrightarrow} \mu_\ell \quad \forall \ell$$

Indeed we have to show that $p_r^{(G_n)}(B_r^{(\mathbb{T}_d^c)}(o_\ell)) \longrightarrow \mu(B_r^{(G)}(o) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell))$ but
$\forall (G, o) \neq \mathbb{T}_d^c(o_\ell),\ \mu((G, o)) = 0$ so $\mu(B_r^{(G)}(o) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell)) = \mu(\mathbb{T}_d^c(o_\ell))$
We take $n > r$ (since we will take the limit as n goes to ∞ anyway) and since

$$p_r^{(G_n)}(B_r^{(\mathbb{T}_d^c)}(o_\ell)) = \frac{1}{\#V(G_n)} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell)\}}$$

we take interest in knowing when $B_r^{(G_n)}(v) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell)$. As we can see in figure 11, this occurs precisely when v is at distance ℓ from the closest leaf. To conclude, we now need to do some counting.

- In $\mathbb{T}_{d,n}$ there are exactly $d(d-1)^{k-1}$ vertices at distance k from the root (since the root has d children (the first generation) and then at every generation 1, ... k-1, each vertex has $d-1$ children).
- Following the same principle, swimming over $k \in [n]$ we have that:

$$\#V(\mathbb{T}_{d,n}) = 1 + d + d(d-1) + ... + d(d-1)^{n-1}$$

.

- For large n we thus have :

$$
\begin{aligned}
\#V(\mathbb{T}_{d,n}) &= 1 + d + d(d-1) + \ldots + d(d-1)^{n-1} = 1 + d\sum_{k=0}^{n-1}(d-1)^k \\
&= 1 + d\frac{1-(d-1)^n}{1-(d-1)} = 1 + d\frac{(d-1)^n - 1}{d-2} = 1 + \frac{d(d-1)^n}{d-2} - \frac{d}{d-2} \\
&= \frac{d(d-1)^n}{d-2} - \frac{2}{d-2} = \frac{d(d-1)^n}{d-2}\left(1 - \frac{2}{d(d-1)^n}\right) = \frac{d(d-1)^n}{d-2}(1+\mathrm{o}(1))
\end{aligned}
$$

- Moreover, being at distance ℓ from the closest leaf in $\mathbb{T}_{d,n}$ is the same as being at distance $k = n - \ell$ from the root. Thus $\mathbb{1}_{B_r^{(G_n)}(v) \simeq B_r^{(\mathbb{T}_d^c)}(o_\ell)} = 1$ for exactly $d(d-1)^{n-\ell-1}$ vertices v. Therefore,

$$p_r^{(G_n)}(B_r^{(\mathbb{T}_d^c)}(o_\ell)) = \frac{d(d-1)^{n-\ell-1}}{\frac{d(d-1)^n}{d-2}(1+\circ(1))} \underset{n\to\infty}{\longrightarrow} (d-2)(d-1)^{-(\ell+1)} = \mu_\ell$$

Hence the result.

Remark:

We see that $\mu \sim \mathcal{G}_0\left(\frac{d-2}{d-1}\right)$. This is not surprising since (see on figure 9) the number of vertices in each generation is $(d-1)$ the number of vertices in the previous one. So each generation has $(d-1)$ times less vertices in $\mathbb{T}_d^c$ when we start at the root leaf.

III.3 Local weak convergence of marked rooted graphs

Definition III.3.1 ***Marked Graph***

A marked (multi-)graph is a (multi-)graph $G = (V(G), E(G))$ together with a set $M(G)$ of marks taking values in a complete separable metric space Ξ called the mark space. Here M maps from $V(G)$ and $E(G)$ to Ξ. Images are called marks. Each edge is given two marks, one for each endpoints. Thus we could think of the marks as located at the half edges incident to a vertex. Such a half-edge can be formalized as $(u, \{u,v\})$ for $u \in V(G)$ and $\{u,v\} \in E(G)$. Alternatively, we can think of the marks as located on a directed edge (u,v). As usual we assume that the degrees are locally finite. We denote the marked rooted graph by $(G, o, M(G)) = ((V(G), E(G)), o, M(G))$.

Definition III.3.2 ***Metric on marked rooted graph***

Let d_Ξ be a metric on the space of marks Ξ. Then, we let:

$d_{loc}((G_1, o_1, M_1(G_1)), (G_2, o_2, M_2(G_2))) = \frac{1}{1+R^\star}$ where:

$$
\begin{aligned}
R^\star = \sup\{r :\ & \exists\phi \text{ an isomorphism between } B_r^{(G_1)}(o_1) \text{ and } B_r^{(G_2)}(o_2) \text{ such that} \\
& d_\Xi(m_1(u), m_2(\phi(u))) \leqslant \frac{1}{r} \forall u \in V(B_r^{(G_1)}(o_1)) \text{ and } d_\Xi(m_1(u,v), m_2(\phi(u,v))) \leqslant \frac{1}{r} \\
& \forall \{u,v\} \in E(B_r^{(G_1)}(o_1))\}
\end{aligned}
$$

Remark:

When Ξ is a finite set, we can take $d_\Xi(a, b) = \mathbb{1}_{\{a \neq b\}}$ so that the above condition states that the graph must be isomorphic but also that the marks must be equal.

IV Local convergence of random graphs

IV.1 Definition and criterion

Definition IV.1.1 ***Local Convergence of Random Graphs***

Let $(G_n)_{n\in\mathbb{N}}$ with $G_n = (V(G_n), E(G_n))$ denote a sequence of (possibly disconnected) finite random graphs. Then:

1. G_n converges locally weakly to $(\bar{G}, \bar{o})$ having law $\bar{\mu}$ (deterministic) when for every continuous bounded function $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$

$$\mathbb{E}[h(G_n, o_n)] \longrightarrow \mathbb{E}_{\bar{\mu}}[h(\bar{G}, \bar{o})]$$

(where the first expectation is with respect to the random vertex **and** the random graph). It is equivalent to $(G_n, o_n) \xrightarrow{d} (G, o)$

2. G_n converges locally in probability to (G, o) having law μ (possibly random) when for every continuous bounded function $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$

$$\mathbb{E}[h(G_n, o_n) \mid G_n] \xrightarrow{\mathbb{P}} \mathbb{E}_\mu[h(G, o)]$$

We write $G_n \xrightarrow{\mathbb{P}} (G, o)$.

3. G_n converges locally almost surely to (G, o) having law μ (possibly random) when for every continuous bounded function $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$

$$\mathbb{E}[h(G_n, o_n) \mid G_n] \xrightarrow{\text{a.s.}} \mathbb{E}_\mu[h(G, o)]$$

We write $G_n \xrightarrow{\text{a.s.}} (G, o)$.

Remark:

- Since μ is a possibly random probability distribution on $\mathcal{G}_\star$ we denote

$$\mu : \left| \begin{array}{ccc} \Omega & \longrightarrow & \mathcal{M}_{\mathbb{P}}(\mathcal{G}_\star) \\ \omega & \longmapsto & \mu(\omega) \end{array} \right.$$

where $\mathcal{M}_{\mathbb{P}}(\mathcal{G}_\star)$ is the space of probability measures on rooted graphs.

- In 2. and 3. we require that $G_n \xrightarrow{\mathbb{P}} (G, o)$ (resp. $G_n \xrightarrow{a.s.} (G, o)$) without including the root o_n drawn u.a.r.

- Notice that for finite graphs, $\mathbb{E}[h(G_n, o_n) \mid G_n] = \frac{1}{\#V(G_n)} \sum_{v\in V(G_n)} h(G_n, v)$ is a deterministic (continuous, thus measurable) function of G_n. Thus, it defines a random variable from an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The same holds for $\mathbb{E}_\mu[h(G, o)] : \omega \mapsto \mathbb{E}_{\mu(\omega)}[h(G, o)]$, where we implicitly assumed in the definition IV.1.1 that μ is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and maps to the space of probability measures on $\mathcal{G}_\star$ endowed with the σ-algebra $\mathcal{S} := \sigma(h_B : \mu \mapsto \mu(B),\ B \in \mathcal{B}(\mathcal{G}_\star))$, where $\mathcal{B}(\mathcal{G}_\star)$ refers to the σ-algebra generated by the open sets (for the local topology) of $\mathcal{G}_\star$ (and μ is indeed $\mathcal{S}$-measurable).

Corollary IV.1.1 ***Relation between local limits***

Assume that $(G_n)_{n\geqslant 1}$ converges locally in probability to (G,o) having law μ. Then $(G_n, o_n) \xrightarrow{d} (\bar{G}, \bar{o})$ having law $\bar{\mu}$ where $\bar{\mu}(.) = \mathbb{E}[\mu(.)]$ is (well-)defined for every $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$ continuous bounded, by:

$$\mathbb{E}_{\bar{\mu}}[h(\bar{G}, \bar{o})] = \mathbb{E}[\mathbb{E}_\mu[h(G,o)]]$$

Moreover, if $(G_n)_{n\geqslant 1}$ converges locally almost surely then it converges locally in probability (towards the same limit).

Proof:

Let $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$ be a continuous and bounded function. Then $\mathbb{E}_\mu[h(G,o)]$ is a bounded random variable and so is $\mathbb{E}[h(G_n, o_n) \mid G_n]$. Therefore, by the dominated convergence theorem we have that :

$$\mathbb{E}[h(G_n, o_n)] = \mathbb{E}[\mathbb{E}[h(G_n, o_n) \mid G_n]] \longrightarrow \mathbb{E}[\mathbb{E}_\mu[h(G,o)]],$$

which proves the first claim.
Moreover, by definition, since almost sure convergence implies convergence in probability we have the second claim.

■

We recall the following standard theorem. See e.g. [Bil08] for a proof.

Theorem IV.1.1 ***Skorokhod's representation theorem***

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on a metric space S such that μ_n converges weakly to some probability measure μ_∞ on S as $n \to \infty$. Suppose also that the support of μ_∞ is separable. Then there exist S-valued random variables X_n defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that the law of X_n is μ_n for all n (including $n = \infty$) and such that $(X_n)_{n\in\mathbb{N}}$ converges to X_∞, $\mathbf{P}$-almost surely.

Theorem IV.1.2 ***Criterion for local convergence of random graphs***

Let $(G_n)_{n\geqslant 1}$ be a sequence of graphs. Then:

1. $(G_n, o_n) \xrightarrow{d} (G, o)$ having law $\bar{\mu}$ precisely when for every rooted graph $H^\star \in \mathcal{G}_\star$ and all integers $r \geqslant 0$:

$$\mathbb{E}[p_r^{(G_n)}(H^\star)] = \mathbb{E}\left[\frac{1}{\#V(G_n)} \sum_{v\in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v)\simeq H^\star\}}\right] \longrightarrow \bar{\mu}(B_r^{(G)}(o) \simeq H^\star). \qquad (2)$$

2. G_n converges locally in probability to (G,o) having law μ precisely when for every rooted graph $H^\star \in \mathcal{G}_\star$ and all integers $r \geqslant 0$:

$$p_r^{(G_n)}(H^\star) = \frac{1}{\#V(G_n)} \sum_{v\in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v)\simeq H^\star\}} \xrightarrow{\mathbb{P}} \mu(B_r^{(G)}(o) \simeq H^\star).$$

3. G_n converges locally almost surely to (G,o) having law μ precisely when for every rooted

graph $H^\star \in \mathcal{G}_\star$ and all integers $r \geqslant 0$:

$$p_r^{(G_n)}(H^\star) = \frac{1}{\#V(G_n)} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H^\star\}} \xrightarrow{a.s.} \mu(B_r^{(G)}(o) \simeq H^\star).$$

Proof:

1. This is exactly as in theorem III.1.2 except for the fact that we take the expectation with respect to the random vertex **and** the random graph by noticing that :

$$\mathbb{E}[h(G_n, o_n)] = \mathbb{E}[\mathbb{E}[h(G_n, o_n)|G_n]] = \mathbb{E}\left[\frac{1}{\#V(G_n)} \sum_{v \in V(G_n)} h(G_n, v)\right]$$

   where the first expectation is wrt G_n **and** o_n and the last one is wrt G_n.

2. The necessary condition is immediate since this is the definition for $h : (G, o) \longmapsto \mathbb{1}_{\{B_r^{(G)}(o) \simeq H^\star\}}$ (which is continuous and bounded as we already saw).
   We now look at the sufficiency part:
   We assume that for all $r \geqslant 0$ and $H^\star \in \mathcal{G}_\star$:

$$p_r^{(G_n)}(H^\star) \xrightarrow{\mathbb{P}} \mu(B_r^{(G)}(o) \simeq H^\star)$$

   Since the set of graphs $\mathcal{H}_\star$ that can occur as r-neighborhoods is countable (modulo isomorphism) we have the convergence in probability of the family of random variables $\{p_r^{(G_n)}(H^\star),\ H^\star \in \mathcal{H}_\star(r)$, where $\mathcal{H}_\star(r)$ is a representative of the class of isomorphism of graphs that can occur as r-neighborhoods. Thus, by *Skorokhod's representation theorem*, there exists a probability space for which this convergence occurs almost surely meaning that we have the almost sure convergence for all such $H^\star \in \mathcal{H}_\star$ and thus for all $H^\star$ (since for every other we have automatically $0 \xrightarrow{a.s} 0$). Then we use point 3. and we have that for random variables defined on this well chosen probability space :

$$\mathbb{E}[h(G_n, o_n) \mid G_n] \xrightarrow{a.s.} \mathbb{E}_\mu[h(G, o)]$$

   Thus,

$$\mathbb{E}[h(G_n, o_n) \mid G_n] \xrightarrow{\mathbb{P}} \mathbb{E}_\mu[h(G, o)]$$

3. The necessary condition is immediate since this is the definition for $h : (G, o) \longmapsto \mathbb{1}_{\{B_r^{(G)}(o) \simeq H^\star\}}$ (which is continuous and bounded as we already saw).
   We now look at the sufficiency part:
   We assume that for all $r \geqslant 0$ and $H^\star \in \mathcal{G}_\star$:

$$p_r^{(G_n)}(H^\star) \xrightarrow{a.s.} \mu(B_r^{(G)}(o) \simeq H^\star)$$

   Then there exists $N \subset \Omega$ a negligible subset such that:

$$\forall r \geqslant 1,\ \forall H^\star \in \mathcal{H}_\star,\ \forall \omega \in \Omega \setminus N,\ p_r^{(G_n(\omega))}(H^\star) \longrightarrow \mu(\omega)(B_r^{(G)}(o) \simeq H^\star)$$

   Then, by theorem III.1.2 we have that for every continuous bounded function $h : \mathcal{G}_\star \longrightarrow \mathbb{R}$ and $\omega \in \Omega \setminus N$

$$\mathbb{E}[h(G_n, o_n) \mid G_n](\omega) \longrightarrow \mathbb{E}_{\mu(\omega)}[h(G, o)]$$

   Thus,

$$\mathbb{E}[h(G_n, o_n) \mid G_n] \xrightarrow{a.s.} \mathbb{E}_\mu[h(G, o)]$$

■

Remark:

If $\#V(G_n) = n$, then the first point of theorem IV.1.2 becomes:

$$\mathbb{E}[p_r^{(G_n)}(H^\star)] = \frac{1}{n} \sum_{v \in [n]} \mathbb{P}(B_r^{(G_n)}(v) \simeq H^\star) \longrightarrow \bar{\mu}(B_r^{(G)}(o) \simeq H^\star).$$

Theorem IV.1.3 ***Local convergence and subsets***

Let $(G_n)_{n \geqslant 1}$ be a sequence of rooted graphs. Let (G, o) be a random variable on $\mathcal{G}_\star$ having law $\bar{\mu}$. Let $\mathcal{T}_\star \subset \mathcal{G}_\star$ be a subset of the space of rooted graphs. Assume that $\bar{\mu}((G, o) \in \mathcal{T}) = 1$. Then $(G_n, o_n) \xrightarrow{d} (G, o)$ when (2) holds for all $H^\star \in \mathcal{T}$ and all $r \geqslant 1$. The same theorem holds for local convergence in probability and local convergence almost surely with $\bar{\mu}$ replaced by μ.

Proof:

First assume that $G_n \xrightarrow{\text{a.s.}} (G, o)$ We remark that for almost all $\omega \in \Omega$ theorem III.1.3 implies that for all $H^\star \notin \mathcal{T}_\star$:

$$p_r^{(G_n(\omega))}(H^\star) \longrightarrow 0 = \mu(\omega)(B_r^{(G)}(o) \simeq H^\star).$$

Thus we have that $p_r^{(G_n)}(H^\star) \xrightarrow{a.s.} 0 = \mu(B_r^{(G)}(o) \simeq H^\star)$ hence the result for almost sure convergence.
The result for local convergence in distribution is obtained exactly as in Theorem III.1.3 except for the fact that we now take the expectation of $p_r^{(G_n)}(H^\star)$ (with respect to the law of G_n).
Finally the result for local convergence in probability is directly obtained by the proof of theorem IV.1.2 since we showed that it was sufficient to prove the convergence in probability of $p_r^{(G_n)}(H^\star)$ for the $H^\star$ that can occur as r-neighborhoods. Thus by copying the argument (*Skorokhod's representation theorem*) we have that it is sufficient to show it for $H^\star \in \mathcal{T}_\star$.

■

IV.2 An example: The local convergence of random regular graphs

Theorem IV.2.1 ***Local convergence of random regular graphs***

Fix $d \geqslant 1$ and assume that nd is even. We denote $UG_n(d)$ the random regular graph of degree d (i.e. the uniform random graph with prescribed degree distribution d) and size n. Then, $UG_n(d)$ converges locally in probability to $\mathbb{T}_d$, the rooted d-regular tree.

Proof:

Let $(G_n)_n$ be a configuration model $CM_n(d)$ (i.e the graph constructed by taking n vertices, each having d half-edges that are then randomly connected together). We will prove the result for G_n and then recover our theorem by conditioning on the graph being simple (i.e having no loops or multiple edges) since the law of $UG_n(d)$ is the same as the law of $CM_n(d)$ conditionned on being simple.
Since the limit we want to achieve is the random d-regular tree each vertex has the same neighborhood. Thus, by theorem IV.1.3 we only need to show that for o a fixed vertex of $\mathbb{T}_d$,

$$p_r^{(G_n)}(B_r^{(\mathbb{T}_d)}(o)) \xrightarrow{\mathbb{P}} 1, \qquad \text{for all } r \geqslant 0$$

Let $r \geqslant 0$ and denote $H^\star = B_r^{(\mathbb{T}_d)}(o)$. In order to prove this claim, we will use a second moment method.

- Step 1: Show that $\mathbb{E}[p_r^{(G_n)}(H^\star)] \longrightarrow 1$.

$$
\begin{aligned}
p_r^{(G_n)}(H^\star) &= \frac{1}{\#V(G_n)} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H^\star\}} \\
&= \frac{1}{n} \sum_{v \in [n]} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H^\star\}} \\
\Longrightarrow 1 - p_r^{(G_n)}(H^\star) &= \frac{1}{n} \sum_{v \in [n]} \mathbb{1}_{\{B_r^{(G_n)}(v) \not\simeq H^\star\}} \\
\Longrightarrow 1 - \mathbb{E}[p_r^{(G_n)}(H^\star)] &= \frac{1}{n} \sum_{v \in [n]} \mathbb{E}[\mathbb{1}_{\{B_r^{(G_n)}(v) \not\simeq H^\star\}}] \\
&= \frac{1}{n} \sum_{v \in [n]} \mathbb{P}(\{B_r^{(G_n)}(v) \not\simeq H^\star\}) \\
&= \mathbb{P}(\{B_r^{(G_n)}(o_n) \not\simeq H^\star\}) \\
&= \mathbb{P}(\{B_r^{(G_n)}(o_n) \text{ is not a tree})
\end{aligned}
$$

(where o_n is a vertex chosen u.a.r.)
For this event to happen, a cycle needs to be present within distance r of o_n. When we take o_n u.a.r we then need, according to the model, to pair its d half edges with (at most) d others vertices. We then repeat the process for the $d-1$ remaining half edeges of the (at most) d vertices we chose. By induction we need to repeat this operation for at most:

$$d + d(d-1) + ... + d(d-1)^r \text{ half edges.}$$

We then repeat the process for each vertex in the r-neighborhood of o_n and there is only a finite number of such vertices. Thus, when $n \longrightarrow \infty$ (meaning the number of vertices goes to infinity) the probability that a cycle is present tends to 0 (because we choose the vertices among a set of cardinality growing to infinity). Thus $\mathbb{E}[p_r^{(G_n)}(H^\star)] \longrightarrow 1$.

- Step 2: Show that $\mathrm{Var}\left(p_r^{(G_n)}(H^\star)\right) \longrightarrow 0$. By step 1. we only need to show that $\mathbb{E}[p_r^{(G_n)}(H^\star)^2] \longrightarrow 1$

$$
\begin{aligned}
\mathbb{E}[p_r^{(G_n)}(H^\star)^2] &= \frac{1}{n^2}\left(\sum_{v \in [n]} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H^\star\}} + \sum_{1 \leqslant u \neq v \leqslant n} \mathbb{1}_{\{B_r^{(G_n)}(u) \simeq H^\star\} \bigcap \{B_r^{(G_n)}(v) \simeq H^\star\}}\right) \\
&= \frac{1}{n^2} \sum_{1 \leqslant u,v \leqslant n} \mathbb{1}_{\{B_r^{(G_n)}(u) \simeq H^\star\} \bigcap \{B_r^{(G_n)}(v) \simeq H^\star\}} \\
\Longrightarrow 1 - \mathbb{E}[p_r^{(G_n)}(H^\star)^2] &= \mathbb{E}\left[\frac{1}{n^2} \sum_{1 \leqslant u,v \leqslant n} \mathbb{1}_{\{B_r^{(G_n)}(u) \not\simeq H^\star\} \bigcup \{B_r^{(G_n)}(v) \not\simeq H^\star\}}\right] \\
&= \frac{1}{n^2} \sum_{1 \leqslant u,v \leqslant n} \mathbb{P}(B_r^{(G_n)}(u) \not\simeq H^\star \text{ or } B_r^{(G_n)}(v) \not\simeq H^\star) \\
&= \mathbb{P}(B_r^{(G_n)}(o_n^1) \not\simeq H^\star \text{ or } B_r^{(G_n)}(o_n^2) \not\simeq H^\star) \\
&= \mathbb{P}(B_r^{(G_n)}(o_n^1) \text{ or } B_r^{(G_n)}(o_n^2) \text{ is not a tree})
\end{aligned}
$$

(where o_n^1 and o_n^2 are two independent vertices in $[n]$ chosen u.a.r.)
As before this quantity goes to 0 when n goes to ∞ (using the same argument but for two graphs).

- Step 3: Conclusion for configuration model.
  By Tchebychev's inequality, and using the trick
  $p_r^{(G_n)}(H^\star) - 1 = (p_r^{(G_n)}(H^\star) - \mathbb{E}[p_r^{(G_n)}(H^\star)]) + (\mathbb{E}[p_r^{(G_n)}(H^\star) - 1])$, we have that

$$\begin{aligned}\forall \epsilon > 0, \quad \mathbb{P}(|p_r^{(G_n)}(H^\star) - 1| \geqslant \epsilon) &\leqslant \frac{1}{\epsilon^2}\mathbb{E}[(p_r^{(G_n)}(H^\star) - 1)^2] \\ &= \frac{1}{\epsilon^2}\left(\mathrm{Var}(p_r^{(G_n)}(H^\star)) + (\mathbb{E}[p_r^{(G_n)}(H^\star) - 1])^2\right)\end{aligned}$$

  Both terms goes to 0 when n goes to ∞ as we saw in steps 1. and 2. hence the result.

- Step 4: General conclusion
  As we said, a (uniform) random graph can be seen as a graph obtained throught the configuration model by conditioning it to being simple. We then use the following admitted result (see e.g. [Hof16] for a proof):

$$\lim_{n\to\infty} \mathbb{P}(CM(d) \text{ is a simple graph}) = C > 0$$

  Thus for all $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(|p_r^{(UG_n(d))}(H^\star) - 1| \geqslant \epsilon) &= \mathbb{P}(|p_r^{(CM_n(d))}(H^\star) - 1| \geqslant \epsilon \mid CM_n(d) \text{ is simple}) \\ &= \frac{\mathbb{P}(|p_r^{(CM_n(d))}(H^\star) - 1| \geqslant \epsilon,\ CM_n(d) \text{ is simple})}{\mathbb{P}(CM_n(d) \text{ is simple})} \\ &\leqslant \frac{\mathbb{P}(|p_r^{(CM_n(d))}(H^\star) - 1| \geqslant \epsilon)}{\mathbb{P}(CM_n(d) \text{ is simple})} \\ &\underset{n\to\infty}{\longrightarrow} 0 \qquad \text{(using step 3. and the previous admitted result)}\end{aligned}$$

  Thus, we showed that $UG_n(d)$ converges locally in probability towards $\mathbb{T}_d$.

■

Remark:

The steps 2 and 3 in the previous proof are unnecessary since $p_r^{(G_n)}(H^\star)$ is a random variable bouded by 1. Indeed, step 1 thus implies that $p_r^{(G_n)}(H^\star) \xrightarrow{L^1} 1$ (by positivity). This directly implies that $p_r^{(G_n)}(H^\star) \xrightarrow{\mathbb{P}} 1$. However, since the second moment could be applied in similar contexts, it was still interesting to see it.

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